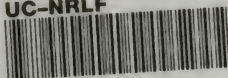


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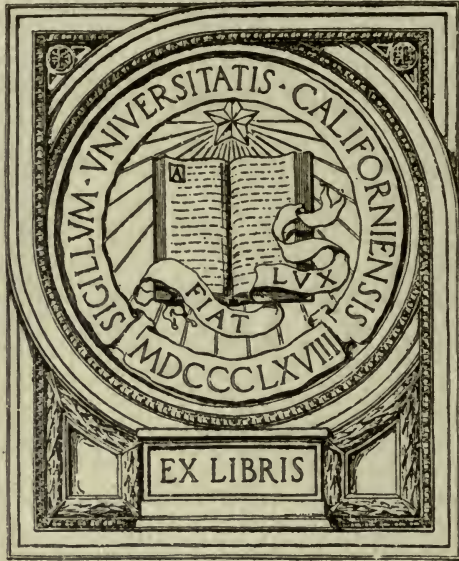
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Final Report
OF THE
National Committee of Fifteen
ON
Geometry Syllabus

July, 1912

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FINAL REPORT OF THE NATIONAL COMMITTEE OF FIFTEEN ON GEOMETRY SYLLABUS

SECTION A. HISTORICAL INTRODUCTION

The committee regards the historical statement printed in the volume of *Proceedings* for 1911, pp. 607-35 inclusive, and constituting Section A of the final report, as most important and calls attention to the following additional reference to this section:

The committee recommends the foregoing historical sketch to the careful consideration of teachers of geometry. Special attention is called to the age-long contest between the extreme formalists and the extreme utilitarians. The committee stands for neither extreme position. It recommends a reasonable attention to exercises in concrete setting, such, for instance, as simple problems involving the trigonometric ratios in connection with similar triangles, or such applications as those shown on pages 21-24 of this report. But in so doing it does not recommend diminishing attention to the logical side of the subject, but rather a quickening of the logical sense thru a more rational distribution of *emphasis* which will make for economy of both time and mental energy in mastering standard theorems and leave opportunity for a broader view of the subject in its concrete relations. See Sections D and E.

SECTION B. LOGICAL CONSIDERATIONS

AXIOMS

(a) **Nomenclature.**—The best historical usage distinguishes between Axioms (Euclid's "Common Notions") and Postulates (Euclid's *aitemata* or "requests") by including in the former certain general statements assumed for all mathematics, and in the latter certain specifically geometric concessions. These names and this distinction are now in general use and there seems no good reason for attempting to change them. However, teachers who may wish to use the single term "assumptions" to cover both, or to use the term "axiom" to mean any proposition whose truth is postulated, thus making axiom and postulate synonymous, should be free to do so.

(b) **General Nature.**—It is evident that strict mathematical science would lead us to seek and to recommend an "irreducible minimum" of assumptions, while educational science leads us to see that such a list would be unintelligible to pupils and therefore unusable in the schools. Since we

cannot recommend the adoption of a set of assumptions along the Hilbert line, we therefore lay down the general line of axioms and postulates needed in geometry, without insisting upon an exact list or upon any particular phraseology.

(c) General List of Axioms.—

As to the nature of the quantities, positive quantities are to be understood. When the negative quantity enters into elementary geometry it is in the discussion of propositions and not in cases in which the axioms are directly employed. For example, it is desirable not to confuse beginners in geometry by the question of dividing unequals by negative numbers.

Operations upon equal quantities.—It should be stated, preferably in a series of axioms, that if equals are operated upon by equals in the same way, the results are equal: i.e., if $a=b$ and $x=y$, then $a+x=b+y$, $a-x=b-y$ (where $a>x$), $ax=by$, etc.

Operations upon unequal quantities.—It should be stated that if unequals are operated on by equals in the same way, the results are unequal in the same order; i.e., if $a>b$ and $x=y$, then $a+x>b+y$, etc. These various cases enter into elementary geometry, and this assumption should be stated in such a manner that the student can easily refer to it in his work.

There is also the assumption that if unequals are added to unequals in the same order, the sums are unequal in the same order, and that if unequals are subtracted from equals the remainders are unequal in the reverse order, these being the only ones relating to inequalities that are needed in elementary geometry.

As to substitutions.—In geometry it is continually necessary to make use of the assumption that a quantity may be substituted for its equal in an equation or in an inequality. Often this assumes the common form that “quantities that are equal to the same quantity are equal to each other.” The committee recommends this axiom.

Inequality among three quantities.—It is necessary to say in geometry that if $a>b$ and $b>c$ then $a>c$, and an axiom to this effect is necessary.

The whole and its part.—Altho the definition of “whole” might be given in such a manner as to render unnecessary the usual axiom, it seems advisable to make the statement in the ordinary form.

It is to be understood that in applying these axioms to geometric magnitudes, the letters used refer to the *numerical measures* of such magnitudes, and as such belong to arithmetic and algebra, thus giving the theoretical basis for the correlation of geometry with these subjects.

Summary.—Axioms covering the above points are of advantage in the practical teaching of geometry, but this committee has no recommendation to make as to order or phraseology. They may be summarized as follows:

If $a=b$ and $x=y$, then

(1) $a+x=b+y$.

(2) $a-x=b-y$ ($a>x$).

(3) $ax=by$.

(4) $a/x=b/y$.

(5) $a^n = b^n$ and $\sqrt[n]{a} = \sqrt[n]{b}$, where n is a positive integer.

(6) If $a > b$ and $x = y$, then $a + x > b + y$, $ax > by$, $a/x > b/y$, and if $a > x$, then $a - x > b - y$.

(7) If $a > b$ and $c > d$, then $a + c > b + d$, and if $x = y$, then $x - a < y - b$.

(8) If $a = x$, and $b = x$, then $a = b$.

(9) If $x = a$, we may substitute a for x in an equation or in an inequality.

(10) If $a > b$ and if $b > c$, then $a > c$.

(11) The whole is greater than any of its parts, and is equal to the sum of all its parts.

(d) **General List of Postulates.**—

(1) *One straight line and only one can be drawn thru two given points.*

Corollary 1. *Two points determine a straight line.*

Corollary 2. *Two straight lines can intersect in only one point.*

(2) *A straight line-segment may be produced to any required length.*

This includes one postulate and one problem of Euclid, and so manifestly depends upon the simplest uses of straight edge and compasses as to be a proper geometric assumption.

(3) *A straight line is the shortest line between two points.*

(4) *A circle may be described with any given point as a center and any given line-segment as a radius.*

(5) *Any figure may be moved from one place to another, without altering its size or shape.*

(6) *All straight angles are equal.*

This and the following corollaries may be included among the theorems for informal proof under Section E.

Corollary 1. *All right angles are equal.*

Corollary 2. *From a point in a line only one perpendicular can be drawn to the line.*

Corollary 3. *Equal angles have equal complements, equal supplements, and equal conjugates.*

Corollary 4. *The greater of two angles has the less complement, the less supplement, and the less conjugate.*

The above axioms and postulates may be recommended for use as soon as the formal proof of propositions is begun, the postulate of parallels being introduced when needed, as follows:

Postulate of Parallels. *Thru a given point one line and only one can be drawn parallel to a given line.*

The question of limits is considered later. It is not deemed desirable to postulate explicitly the existence of such concepts as point, line, and angle, nor to assume that a line drawn thru a point in a triangle must cut the perimeter twice, nor to add a postulate of continuity. It is well, however, for teachers to mention that such assumptions are always tacitly made.

In any case the committee feels that a certain amount of care should be

taken in fixing the location of points and lines and proving that lines intersect, when the accuracy of the proof in question might be affected by ignoring such details.

DEFINITIONS

(a) New Terms.—

(1) *General principle*.—It is unwise for individual teachers or writers to introduce terms beyond those actually in common use in geometry, or to change the accepted meaning of common terms, unless there seems to be a very definite advantage in the new term and an unquestionable sanction in the mathematical world. In particular, the substitution of a new term for an old one to denote the same concept is undesirable.

(2) *Type of terms that may safely be added* to those of the older elementary geometry.—*Congruent*, because this is so widely used both here and abroad, and because it avoids the loose use of *equal* and the long forms of *identically equal*, and *equal in all their parts*.

(3) *Types of terms that may safely be dropped*.—*Scholium*, because this has been so generally abandoned, and because it is unnecessary; *mixed line*, an antiquated term of no value in elementary geometry. Other such terms are *trapezium* and *rhomboid*.

(4) *Types of terms that seem of too doubtful advantage* to be recommended definitely by this committee, teachers being left free to use them if they desire, the terms thus being given an opportunity to make their way if they possess real merit.—*Ray*, a term that has abundant sanction in higher geometry, but may be dispensed with in elementary work. Other such terms are *mid-join* (for median), *cuboid* (for rectangular parallelopiped), *n-gon* (for polygon of *n* sides), and *sect* (for segment of a straight line).

(5) *Types of terms that are used with a different meaning in higher geometry*, and that may properly be used in elementary geometry with the more recent signification.—*Circle* as meaning the line, which is, indeed, the primitive Greek meaning; *circumference*, as meaning the length of the circle—these usages requiring a re-defining of *segment of circle*, *semicircle*, *area of circle*, and other obvious terms.

The committee recognizes also the tendency to unify the usage of such terms as *polygon* and *sphere* in elementary and higher geometry. This tendency should be encouraged. In any case, it is essential that the pupil should understand clearly what the terms mean in the statements and proofs of the propositions.

(b) Symbols.—

(1) *General principle*.—No symbols should be recommended beyond such as are already in wide use in elementary geometry, and any that are unnecessary or are not generally accepted should be abandoned. The elaboration of personal symbolism, sometimes to the point of eccentricity, is such as to be cumbersome in the mathematics of the present.

(2) *Recommendations*.—The committee feels that the common symbols of algebra, most of which are known to the pupil beginning geometry, and such obvious symbols as those for perpendicular, triangle, circle, square, and parallel, are all that are needed in a course in elementary geometry, and that it is unnecessary to specify these symbols in detail. It appears that there is no generally received symbol for congruence, the symbols $=$, $\equiv$, and $\cong$ all being in use, and it seems best to recognize this fact, leaving teachers at present to decide the question for themselves. In due time a general consensus of opinion may lead to some definite usage, and it is the feeling of the committee that the second symbol given is a desirable one.

(c) *Distribution of Definitions*.—The committee recommends that new terms be taught when the time arrives for using them. This allows a teacher to use a book in which the definitions are massed or one in which they are scattered, but it encourages teaching them on the latter plan. It is recognized that the massed plan has the advantage of a dictionary arrangement, and this is a plan that a textbook writer might reasonably adopt, but it is not a plan to be followed in the actual teaching of the terms.

(d) *The Defined and the Undefined*.—The attention of teachers is called to the fact, now coming to be well recognized, that certain terms in geometry must be looked upon as *undefined*.

Certain concepts are so elementary that no simpler terms exist by which to define them, altho they can easily be explained. For example, *point*, *line*, *surface*, *space*, *angle*, *straight line*, *curve*. The committee recommends that teachers give more attention to instilling a clear concept of such terms and none to exact definition.

On the other hand, the committee recommends the careful definition of readily definable terms, where these definitions are parts of subsequent proofs, such definitions to be memorized exactly or in their essentials. For example, *right angle*, *square*, *isosceles triangle*, *parallelogram*.

There is a further class of easily defined terms, where the definition is not made the basis of a proof, and it seems obvious to the committee that the memorizing of the exact wording of such definitions is not a wise expenditure of time. Such terms are *hexagon*, *heptagon*, *reëntrant angle*, *concave polygon*, etc.

(e) *The Form of Definition*.—A definition may begin with the term defined, as in a dictionary; or it may close with the word defined; or it may at times contain the word in the midst of the sentence. The committee feels that it is of no moment which of these forms is taken, or that the definition be embodied in a single sentence. A definition that is to be memorized as the basis of a proof should be as nearly scientific as the powers of a beginner in geometry will justify, containing only terms that are

simpler than the term defined, not being tautological, and being reversible—but further than this it seems unwise to attempt to specify the form of a definition.

INFORMAL PROOFS

(a) **Justification.**—It is not pretended that elementary geometry is a perfect piece of logic. In general, the modern departures from Euclid have sacrificed logic for other ends, and even Euclid's *Elements* was not without numerous logical imperfections. That is to say, it has always been considered justifiable to sacrifice logic to a greater or less degree. The principle is that a logical sequence should be maintained, and formal proofs of propositions necessary to the sequence should be required, so far as this is consonant with the educational principle of adapting the matter to the mind of the learner. Now in many cases it happens that an informal and confessedly incomplete proof is more convincing to a beginner than a formal and complete one, and is less discouraging because it postpones the minor and seemingly unimportant steps to a time when their importance may be appreciated and the proofs understood.

(b) **Types.**—To be specific, the following are types of propositions that are better passed over by the beginner without a formal statement, being introduced at the proper points in the development, or with informal proof, than proved in the Euclidean fashion:

If one straight line meets another the sum of the two adjacent angles is a straight angle, and conversely (and related propositions);

All straight angles are equal (a proper postulate with related corollaries);

Two straight lines can intersect in only one point;

A straight line can have but one point of bisection (and the related case for angles);

The bisectors of vertical angles lie in one straight line;

Polygons similar to the same polygon are similar to each other;

If one angle is greater than another, its complement is less than the complement of the other (and related propositions);

A straight line can cut a circle (circumference) in two points only;

Circles of equal radii are equal (and related statements);

All radii of the same circle are equal (and similarly for diameters);

A circle can have but one center;

And propositions relating to the conditions under which two circles (circumferences) intersect.

It should be understood that these propositions are merely types, and that others of the same type may be treated in the same way, as specified in Section E of this report.

(c) **Experience of Other Countries.**—It is the experience of all countries where Euclid is not taught that good results follow from the use of a reason-

able number of such informal proofs. The German and Austrian textbooks are especially given to such procedure, and the results seem to have been favorable rather than otherwise. The number of propositions formally proved in a German textbook is notably less, for example, than in a corresponding French textbook.

(d) **Dangers.**—It is evident, however, that we may easily go to a dangerous and ridiculous extreme in this matter. With all of the experiments at improving Euclid the world has really accomplished very little except as to the phraseology of propositions and proofs; the standard propositions remain, and if geometry has any justification, apart from its kindergarten aspect (which requires but a short time), most of these propositions will continue to be proved, and should continue to be proved. These propositions, whether in the Euclid or Legendre arrangement, number in the neighborhood of 160 for plane geometry. Of this number upward of one hundred must receive formal proof in any well-regulated course in geometry.

TREATMENT OF LIMITS AND INCOMMENSURABLES

(a) **Present Status.**—It is generally agreed that the present treatment of this subject is open to two objections: (1) it is not sufficiently understood by the student to make it worth the while, and (2) it is not scientifically sound.

(b) **Remedies Proposed.**—Corresponding to the two defects mentioned two remedies have been proposed: (1) to make it less formal and technical, so that it shall be better understood, and (2) to abandon the incommensurable case altogether in secondary education.

(c) **The Position of This Committee.**—This committee recommends that in elementary geometry the nature of incommensurables and limits be explained, but that the subject no longer be required for entrance to college or be included in official examinations. It recommends that the schools treat the subject as fully beyond this point as circumstances seem to demand, and to this end reference is made to the syllabus given in Section E.

The prime object is to relieve the schools of the necessity of teaching the subject, while leaving them free to do so if they wish.

TIME AND PLACE IN THE CURRICULUM

(a) **Conventionally.**—At present, in America, plane geometry is generally taught in the tenth school year (not counting the kindergarten).

In the East it is completed in the eleventh school year, and in the West solid geometry is completed in the eleventh or twelfth year. In spite of all the discussion about constructive geometry (intuitive, metrical, etc.) in the first eight grades, carried on in the past half-century, no generally

accepted plan has been developed to replace the old custom of teaching the most necessary facts of mensuration in connection with arithmetic. We have, therefore, at this time, algebra in the ninth school year, plane geometry in the tenth, and algebra and geometry in the eleventh and sometimes in the twelfth.

(b) **Changes Suggested.**—Certain changes in this conventional plan have been suggested.

(1) To provide for preliminary (inductive, constructive, observational) work in geometry in the elementary grades. This topic is discussed in Section C of this report.

(2) To precede the work in plane geometry by some definite work in geometric drawing. Attention may be called to the fact that the recent great advance in art education has had one disadvantage from the standpoint of geometry, in that geometric drawing has been abandoned, and that therefore some little work in handling compasses and ruler must now form part of the first steps in this subject.

(3) To unite geometry and algebra, or geometry and trigonometry. This committee does not feel that the experiments along this line, which have been made in only a few schools, have been sufficient to determine whether or not geometry should run parallel with algebra in the ninth, tenth, and eleventh school years.

(c) **Position of This Committee.**—This committee recommends that plane geometry be assigned not less than one year nor more than one and one-half years in the curriculum, being preceded by at least one year of algebra except where the individual teacher desires to carry it along with algebra.

It should be distinctly understood that owing to the condition of unrest in the entire field of secondary education it is at present impossible to give any final advice along any of these lines of change. It is probable that many of the readjustments now under general discussion will influence every high-school curriculum in the course of time. It is also possible that some of the proposed changes will be adapted by the different types of secondary schools to their own needs, and that they will receive greatly varying emphasis in different localities. A certain amount of experimentation will undoubtedly be necessary to test the feasibility of some of the proposed plans. Great care should be taken to make all such experimentation with due regard for all that was good in the past, so that the new curricula may be the result of evolution and not of revolution.

The most noteworthy tendency in secondary education is the desire for more organic teaching and hence the desire for more time. This tendency finds its most significant expression in the movement toward a *six-year curriculum*. It is undoubtedly true that in a six-year curriculum many of the problems of correlation would be brought nearer to a solution,

that many difficulties arising from the present tandem system would disappear, and that mathematics would be given a place in the curriculum more nearly commensurate with its importance.

For a brief account of the six-year curriculum the reader is referred to the book of Hanus entitled *A Modern School*, published by the Macmillan Company, and also to the *Proceedings of the National Education Association* for 1908.

PURPOSE IN THE STUDY OF GEOMETRY

(a) **Historical Review.**—Geometry was originally, as its name indicates, purely a practical subject. This phase of its history remains in the work in mensuration in arithmetic today. It then became a philosophical subject, connecting with mysticism in the Pythagorean school, being put upon a more solid scientific basis by the Platonists, and being crystallized by Euclid about 300 B.C. Since that time the formal side has dominated. But this formal side has been attacked time after time, by the astrologers and mystics, by the cathedral builders of the Middle Ages, strongly by the French writers of the seventeenth and eighteenth centuries, recently by an extreme school in England, and at present in a less formidable fashion in our own country. The results of these attacks in so far as they have meant the abandoning of formal proofs have been futile.

(b) **The Practical Side.**—In the high school geometry has long been taught because of its mind-training value only. This exclusive attention to the disciplinary side may be fascinating to mature minds, but in the case of young pupils it may lead to a dull formalism which is unfortunate. On the other hand, those who are advocating only a nominal amount of formal proof, devoting their time chiefly to industrial applications, are even more at fault. The committee feels that a judicious fusion of theoretical and applied work, a fusion dictated by common-sense and free from radicalism in either direction, is necessary.

As to the nature of the applications, the committee feels that there are several types of genuine problems, but that many of the so-called real applications either are too technical to be within the grasp of the young beginner, or represent methods of procedure that would not be followed in real life. Moreover, it should be remembered that the very limited time devoted to plane geometry (usually a single year) renders it impracticable to introduce many of the applications that might be desirable if the time were not so restricted.

(c) **The Formal Side.**—No reference to the applications of geometry is to be construed to mean that the committee feels that the formal side should suffer, or that geometry is wanting in a distinct disciplinary value. A formal treatment of geometry, to about the traditional extent, is necessary purely as a prerequisite to the study of more advanced mathematics, and

still more because such treatment has a genuine culture value, for example, in assisting to form correct habits in the use of English.

Certain writers on education have claimed that geometry has no distinctive disciplinary value, or that the formal side is so intangible that algebra and geometry should be fused into a single subject (not merely taught parallel to each other), which subject should occupy a single year and be purely utilitarian. These writers fail to recognize the fundamental significance of mathematics in either its intellectual or its material bearing.

(d) **Claims for Geometry.**—Among the claims in behalf of geometry the committee would emphasize the following:

Geometry is taught because of the *pleasure* it gives when properly presented to the average mind.

Geometry is taught because of the *profit* it gives when properly presented. For example:

(1) It is an exercise in logic, and in types of logic not generally met in other subjects of the school course, and yet types which occur in geometry in unusually simple setting and which are easily carried over into the actual affairs of life. Closely connected with the logical element is the training in accurate and precise thought and expression and the mental experience and contact with exact truth.

This logic may be no more practical than literature or art or any other great branch of learning, but its general effect on the human mind has been doubted by such a small number of scholars as to render it worthy of the highest confidence.

(2) The study of geometry leads also to an appreciation of the dependence of one geometric magnitude upon another, in other words to the tangible concept of *functionality*.

(3) The study of geometry cultivates space intuition and an appreciation of, and control over, forms existing in the material world, which can be secured from no other topic in the high-school curriculum.

(4) The value of the applications of geometry to mensuration and the satisfaction derived by the pupil in verifying the formulas of mensuration already met by him in arithmetic are well recognized by all teachers.

If we had to justify the position of any other subject in the curriculum, history, rhetoric, geography, biology, etc., it is doubtful whether equally specific and cogent reasons could be found. If we were to dismiss geometry with a few practical lessons, much more should we be compelled to dismiss most other subjects in the curriculum with the same treatment.

HISTORICAL NOTES

Of the stimulating effect of occasional bits of historical information given by the teacher or the textbook there can be no question. There is

plenty of material to be found in the well-known elementary histories of the subject. The discoverers of particular propositions are known in a few cases, and the general story of the subject, told informally as the pupil proceeds in his study, adds a human interest that is valuable. Portraits of famous mathematicians may be recommended for the schoolroom.

POINTS RELATING TO SOLID GEOMETRY

(a) **Axioms and Postulates.**—The list of axioms already given need not be increased, but the following postulates may be added:

(1) *One plane and only one can be passed thru two intersecting straight lines.*

Corollary. *A plane is determined by three points not in the same straight line, by a straight line and a point not in it, or by two parallel lines.*

This postulate, which is the analogue of the first postulate in plane geometry, may also be given as a theorem for informal proof.

(2) *Two intersecting planes have at least two points in common.*

(3) *A sphere may be described with any given point as center and any given line-segment as radius.*

It is tacitly assumed that the figures described in the course in solid geometry exist and can be made the subject of investigation; e.g., the prism, pyramid, cylinder, cone, etc.

It may also be assumed, tacitly or explicitly, that the various closed solids have definite areas and volumes; e.g., that a sphere has a definite volume which is less than that of any circumscribed convex polyhedron and greater than that of any inscribed convex polyhedron.

(b) **Definitions.**—Latitude is left to the teacher in regard to the use of such terms as prismatic space, cylindrical space, nappes of a cone, and some of the names suggested for a rectangular parallelopiped, which are convenient but not necessary in an elementary course.

After the analogy of the circle defined as a line, it is proper that the sphere be defined as a surface but the more common definition may be retained if desired.

(c) **Purpose.**—In solid geometry the utilitarian features play an increasingly important part. The mensuration involved in plane geometry is so simple as to be fairly well understood as presented in arithmetic. Solid geometry, however, offers a rather extended field for practical mensuration in connection with algebraic formulas. The subject is therefore particularly valuable for high-school classes. A further application is found in the power afforded to visualize solid forms from flat drawings, a power that is essential to the artisan and valuable to everyone. The committee therefore summarizes the purposes of solid geometry as follows:

(1) To emphasize and continue the values of plane geometry, mentioned above;

(2) To present a reasonable range of applications to the field of mensuration;

(3) To cultivate the power of visualizing solid forms from flat drawings, without entering the technical domain of descriptive geometry.

SECTION C. SPECIAL COURSES

(a) Courses for Different Classes of Students.—

One of the topics which this committee undertook to consider was that of different courses for various classes of students in the high schools.

After investigation, it is the belief of the committee that there should be no attempt to outline such courses. The syllabus as recommended in Section E may be altered in special cases by the omission of the theorems printed in small type and by increased emphasis upon theorems which admit of direct practical applications.

The preceding recommendation, together with the possible omission of solid geometry, would reduce the course to less than half the traditional length. It seems probable that no greater reduction would be desirable even for students in purely commercial courses, or indeed in any course in which formal geometry is a required subject.

(b) Preliminary Courses for Graded Schools.—

A portion of the report of this committee was to deal with preliminary courses to be undertaken in graded schools.

Recommendations.—It is of the utmost importance that some work in geometry be done in the graded schools. For this there are at least two very strong reasons. In the first place, geometric forms certainly enter into the life of every child in the grades. The subject-matter of geometry is therefore particularly suitable for instruction in such schools.

Moreover, the motive for such teaching is direct. The ability to control geometric forms is unquestionably a real need in the life of every individual even as early as the graded school. For those who cannot proceed farther this need is pressing; the direct motive involved compares very favorably with any other direct motive for work in the grades. For those who are going on to the high school, the development of the appreciation of geometric forms is almost an absolute prerequisite for any future work in geometry.

Informal work.—It is quite obvious that no work of formal, logical character should be undertaken in the graded schools. The earliest work in geometry will doubtless be so informal that it will not constitute a separate course. Instruction in drawing, in pattern-making, and in elementary manual training furnishes a basis for considerable geometric work even in the first grades of the primary school.

Such work as this should be encouraged, tho no special outline of it can

be given on account of its dependence upon other courses. The constructions for erecting perpendiculars, bisectors of angles, etc., can and should be given in connection with such manual-training work as making boxes, patterns, etc., tho no technical nomenclature need be used. In such work paper folding and the use of simple instruments should be encouraged, including the compasses, the ruler, and in later years the protractor and squared paper.

Mensuration.—In connection with arithmetic much geometric work may be taken up which is consistent with the child's real interests and life. Measurements may be introduced very early and the mensuration of simple forms such as the square, rectangle, and triangle need not be long delayed. After this, other geometrical forms and solids may be introduced under the head of mensuration even earlier than is now customary. In properly conducted schools, the students will become familiar at the same time with such figures as the circle, cube, sphere, etc., in manual training and in other elementary courses, such as nature-study, geography, etc.

In the later grades practically all of the simple geometric forms will find their place in arithmetic under the head of mensuration, in drawing, and in manual training.

Work in the higher grades.—A special course in geometry in the graded school is desirable, if at all, only in the last grade or the last two grades. In such a course no work of demonstrative character should be undertaken, tho work may be done to *convince* the student of the truth of certain facts; for example, by paper folding or cutting a variety of propositions may be made evident, such as that the sum of the angles of a triangle is 180° , etc.

The theorem just named is typical of the theorems which the student should know as *facts* before he leaves the graded school. Many others of the theorems printed in black-face type in the syllabus submitted herewith may be taught in this course without formal proof.

Theorems as facts.—Emphasis should be laid upon the facts with which the student is already familiar thru the work described above. The course should be regarded partially as a classification and a systematization of the knowledge previously acquired. Thus, simple geometrical forms should be brought up in the connection in which they have arisen in the student's past experience. In taking up constructions, explicit mention should be made of the previous work in which a given construction occurred, and further practical instances of the use of such constructions should be given.

Drawing to scale.—Emphasis should be laid also on other work of a concrete nature which involves direct use of geometric facts. Thus the propositions concerning the similarity of triangles should be introduced by means of the drawing of figures to scale. Attention should be called to the

cases in which figures have been drawn to scale in the past. The usefulness and the necessity of the operation should be emphasized, and such applications as the drawing of house plans, the copying of patterns on a smaller scale, etc., should be given. The use of cross-section paper for this purpose may be encouraged. Finally after the notions involved are very clear indeed, and after actual measurements have been made and reduced to scale, the precise facts regarding similar triangles may be given. This should follow and not precede the work described above. The applications to elementary surveying should be made, if possible, in actual field work.

Models and patterns.—The situation just described for similar triangles should be carried out as far as possible in other instances. Thus the important theorems on the measurement of angles can be illustrated in many ways. The Pythagorean theorem, without formal proof, can be illustrated and made real to every student by reference to the pattern forms in which it occurs, the calculation of distances, and other real applications. Finally the mensuration formulas can all be given. In the latter, concrete illustrations should abound and verification by means of models and measurements upon them should be encouraged.

Forms of solid geometry.—Contrary to the traditional procedure, the forms of solid geometry should be emphasized even more than those of plane geometry, for they are more real and more capable of concrete illustration.

Not only should formal demonstration be avoided but also long lists of definitions which tend to confuse rather than enlighten. Definitions should be stated formally only *after* the concept is clearly found in the student's mind. No axiom should be stated as such at any point, tho frequent assumptions should be made without an attempt at proof. In all such cases care should be taken that the statements made seem reasonable to the student and no forward step should be taken until he is absolutely convinced of the truth of the statement.

Justification.—That such work is of vital value to the student can scarcely be doubted; that it is absolutely legitimate will probably be admitted by all interested in primary education. Its value, its real direct motives, its contact with life, the legitimacy of its subject-matter exceed incomparably those of the traditional course in advanced arithmetic. At least, the course in arithmetic may be vitalized by a liberal infusion of such geometric work.

If such a course is not given in the grades—perhaps even tho it is—a course of similar character but very much shorter may be given in the high school before formal work in demonstrative geometry is attempted. In any event it is desirable that the course in formal geometry should not proceed

in its traditional groove until the teacher is assured that the ideas mentioned above are thoroly familiar to the student.

SECTION D. EXERCISES AND PROBLEMS

DISTRIBUTION, GRADING, AND NATURE OF EXERCISES

(a) **Increasing Number of Exercises.**—There has been a growing tendency in the last two decades to increase *abnormally* the number of exercises to be considered by each pupil under the following heads: (1) long lists of additional theorems (beyond the full set usually given in the texts), (2) long lists of problems of construction having at best remote connection with any uses of geometry within reach of the ordinary high-school pupil, (3) long lists of numerical exercises given in the abstract, that is, unrelated to any concrete situation familiar to the pupil or arousing his interest.

To give a single illustration of each:

(1) The squares of two chords drawn from the same point in a circle have the same ratio as the projections of the chords on the diameter drawn from the same point.

(2) To construct a triangle having given the perimeter, one angle, and the altitude from the vertex of the given angle.

(3) Thru a point P in the side AB of a triangle ABC, a line is drawn parallel to BC so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.

(b) **The Distribution of Exercises.**—It is recommended that there should be treated in connection with each theorem such immediate concrete questions and applications as are available, and especially early in the course should such theorems be given as easily lend themselves to this class of exercises.

For example, in a treatment in which the theorems on congruence of triangles are placed early, there is the opportunity to bring in at once the simplest schemes for indirect measurement of heights and distances. Then later as similarity of triangles is taken up, there is the chance to recur to the same problems and let the pupil see how the principle adds power and facility in making indirect measurements. There is thus a progressive development in the facility for solving concrete problems along with the theory.

This principle can be carried out in many different lines. For example, in connection with triangles, circles, and squares, there are many applications immediately available and easily found in tile patterns, window tracery, grillwork, steel ceiling patterns, etc. These afford fine exercises in construction early in the course, and are equally available later in the computation and comparison of areas. When such exercises are given they

should be distributed as far as possible in connection with the theorems used in the construction and comparison of the figures involved.

However, only the simplest uses of the theorems can be shown in the immediate connection, both because of the space occupied by them and the danger of interrupting the continuity of the theorems by too many exercises thrown in between them, and also because most of these applications make use of various different theorems, and hence must come after certain groups of theorems, thus making necessary occasional lists of problems and applications scattered thru the various books, as well as sets of review exercises at the end of each book.

The whole question of distribution is thus to be determined by the relation of the problems and applications to the single theorems or groups of theorems to which they belong. The important question of emphasis in Section E of this report is best brought out by the grouping of many exercises around the basal theorems.

On the basis of distribution we have all extremes in the various texts, including: (1) the purely logical presentation, that is, the continuous chain of theorems with practically no applications in concrete setting in connection with them and almost none at the end of the books; (2) the same as the foregoing, except that the long sets of exercises are placed at the end of each book, where they loom up before the pupil as great tasks to be ground thru, if, indeed, they are not omitted altogether; (3) the psychological presentation in which the more difficult exercises either are postponed to a later part of the course or are omitted altogether, and the easier ones are brought into more immediate connection with the theorems to which they are related.

The time and space made available by the third method of presentation provide an opportunity for the pupil to gain some acquaintance with the uses of the theorems as he proceeds and to become genuinely interested in the development of the subject. The committee strongly recommends this latter method of presentation. In expressing its disapproval of method (2), it is not to be understood that the committee objects to any textbook because it offers a large number of exercises, placed at the end of each book, from which the teacher is to make a selection. The objection to (2) should be clear from reading (3), which the committee approves.

(c) The Grading of Exercises.—Too much cannot be said in favor of a large number of simple cases rather than too many difficult questions, especially early in the course, but also even thruout the secondary course in geometry.

The average high-school pupil is not likely to become adept at proving difficult and abstruse theorems independently or in solving complicated problems. On the other hand, the rank and file are bound to become discouraged and hopelessly lost in the so-called "originals," unless the

grading is carefully done, and steps of difficulty are kept down to a very reasonable lower limit.

The ideal treatment would seem to be: (1) to make a proposition appeal to the pupil as reasonable by simple illustrations, after which should follow the deductive proof; (2) to apply the theorem to more difficult situations, involving problems which the pupil regards as interesting and worth while. It is recognized that this ideal cannot be attained with reference to all the theorems of geometry but it is believed that it can be attained in very many cases; and, wherever this is possible, great interest and incentive are given to the pupil.

As a matter of fact, familiarity with the elementary truths pertaining to angles, parallelograms, and circles, when consistently tried out and seasoned by applications to numerous comparatively simple and interesting geometric forms suggested by figures which abound in concrete setting on every hand within reach of all, is usually of more value to the average pupil (and even to the better pupils) than is the study of a larger number of abstract theorems or problems thru which they are often forced. Nevertheless, for the benefit of the brighter pupils, it is desirable that a few comparatively difficult problems be given, especially at the ends of the various books, or in a supplementary list.

(d) **The Nature of Exercises.**—This topic has been referred to under (a), (b), (c). It is recognized that a fair proportion of the traditional exercises given in abstract setting should find a place in a course in geometry, but the committee believes that, in accordance with the common practice of the past twenty years, this class of exercises has been magnified and extended, especially with reference to the more difficult exercises, beyond the interest and appreciation of the average pupil.

The committee therefore recommends that a judicious selection of a reasonable number of abstract originals be made in order to leave time for an equally reasonable number of problems, particularly those with local coloring, stated in concrete setting.

Since ample lists of abstract originals are within easy reach of all teachers of geometry, it seems unnecessary to supply illustrations of such exercises. But as the teacher must generally depend upon his own initiative to supply problems in concrete setting, it seems desirable to indicate a few sources from which such problems may be obtained. The committee believes that a reasonable number of problems of this character creates an interest in the minds of the pupils that reacts strongly in augmenting their understanding and appreciation of the logical side of the subject. But it is not to be understood that the committee regards these problems as practical in the narrow sense of the word.

SOURCES OF PROBLEMS

(a) **Architecture, Decoration, and Design.**—Industrial design and architectural ornament are replete with details that may be used as a source of supply for geometry problems. These problems are of three kinds: (1) the problems involved in the construction of the figures themselves; (2) the demonstrations necessary to establish numerous relations which are visible to the mathematician and which must occasionally be assumed by designers; (3) problems in computation.

Among the industrial products that involve geometric ornament are tile and mosaic floors, parquetry, linoleum, oilcloth, steel ceilings, ornamental iron, leaded glass, cut glass, and the like. Figures for problems from these sources may be made from the cuts in trade catalogs.

Problems based on architectural ornament are largely from details of Gothic tracery and can be obtained only by a study of the buildings themselves or of the photographs of them that may be seen in architectural libraries. Gothic tracery is found in windows, in ornamental iron, in carved stone and wood on the outside and inside of buildings, on furniture, choir screens, rafters, and the like, that abound in mediæval cathedrals and churches and in their modern imitations.

These problems have distinct advantages. In many cases their comprehension and solution require no technical knowledge beyond the elementary mathematics needed. These designs abound, are largely within the reach of pupils, and their use in the classroom brings before pupils as nothing else can the beauty and widespread application of geometric forms. In them may be found applications of many topics of elementary mathematics and from them may be obtained numerous exercises of all grades, from the simplest to the most complex. By their use it is possible, therefore, to introduce anywhere in the work problems that are within the reach of the average pupil and appeal to him with a minimum of experiment, explanation, discussion, or previous special preparation.

(b) **Problems of Indirect Measurement.**—It should not be considered that the types of applications under (a) are relatively of greater importance than numerous others. Any application that adds interest to the study of rigorous geometry is of value. Of special interest are all simple means of effecting indirect measurements of distances, such, for instance, as the numerous applications of the congruence theorems and the theorems on similarity of triangles. Here the teacher will find much assistance in Principal Stark's *Measuring Instruments of Long Ago*.¹ Again in considering the isosceles triangle, the universal leveling instrument (aside from the spirit level) offers a number of applications. The form is that of an isosceles triangle bisected by a line from the vertex.

Many simple and interesting problems in indirect measurement are made available by the introduction of the trigonometric ratios, *sine*, *cosine*,

¹ *School Science and Mathematics*, Vol. X, pp. 48, 126.

and *tangent*. This can be done as soon as the theorems on similar triangles are known, and the computation by measurement of a two-place table of natural functions at intervals of 5° affords, of itself, a good drill in the application of these theorems, at the same time providing material for solving concrete problems of great interest to young pupils.

It may not be possible to find time for this in connection with the usual course in plane geometry. Schools that devote most of their time and effort to preparing pupils to pass entrance examinations for college would certainly find it difficult to meet any added requirement. In view, however, of the omissions suggested by this committee and the readjustment of emphasis on basal theorems, time may be found, as the experience of an increasing number of teachers has shown. Where this can be done it constitutes an important step in the closer correlation of the subjects in elementary mathematics. In any case only the natural functions should be used, and the applications should be limited to those involving right triangles.

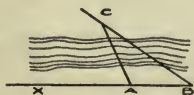
(c) **Other Sources.**—Problems may be obtained also from physics, mechanics, and other sciences, from engineers' and builders' manuals and works on carpentry and masonry, as, for instance, problems derived from various common forms of trusses and the construction of arches. But there is a danger in connection with problems from these sources, that aside from the geometry involved, they may contain technical terms and mechanical terms and mechanical features unknown to the average pupil and not easily understood without more explanation and consequent distraction from the geometry itself than is warranted in the ordinary course. Problems of this class should be carefully tried out and sifted before being adopted for use.

(d) **Illustrative Problems.**—There are given below a few typical problems which are suggested for the purpose of making clear what the committee has in mind. Thru simple problems of these types and many others which might be suggested much interest can be imparted to the study of demonstrative geometry, even tho the problems be not practical in the strict sense of the word. The danger of using too many problems in any narrow field is, however, apparent.

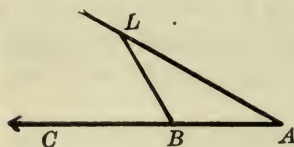
(1) The theorem regarding the angle sum in a triangle has a large number of applications. For example, to measure PC, stand at some convenient point A and sight along APC and (by the help of an equilateral triangle cut from pasteboard) along AB. Then walk along AB until a point B is reached from which BC makes with BA an angle of the equilateral triangle (60°). Then $AC=AB$, and since AP can be measured we can find PC. This is an example of a problem that adds interest to the work without being itself a practical application that would be used by a surveyor.



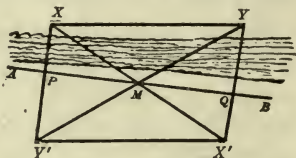
(2) A problem of the same nature is the following: To measure AC, first measure the angle CAX, either in degrees with a protractor or by sighting across a piece of paper and marking it down. Then walk along XA produced until a point B is reached, from which BC makes with BA an angle equal to half of angle CAX. Then it is easily shown that $AB=AC$.



- (3) The sailor makes use of this principle when he "doubles the angle on the bow" to find his distance from a lighthouse or promontory. If he is sailing on the course ABC and he notes a lighthouse L when he is at A, and takes the angle A, and if he notices when the angle that the lighthouse makes with his course is just twice the angle noted at A, then $BL = AB$. He has AB from his log, so he knows the distance BL.



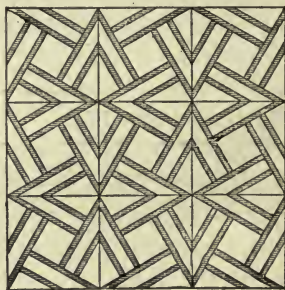
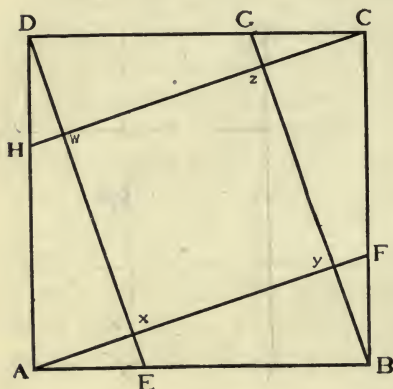
- (4) To measure the line XY, when the observer is at A, we may measure any line AB along the stream. Then the observer may take a carpenter's square, or even a large book, and walk along AB until a point P is reached from which X and B can be seen along two sides of the square. Similarly the point Q may be fixed. Then by walking along YM to a point Y' that is exactly in line with M and Y and also with P and X, the point Y' is fixed. Similarly X' is fixed. Then $X'Y' = XY$.



- (5) A field containing 9 acres is represented by a triangular plan whose sides are 12 in., 17 in., and 25 in. On what scale is the plan drawn?—Conant.

- (6) Assuming the earth to be a sphere of which the radius is 3,960 miles, find the length of one degree of longitude at 60° north latitude, and compare its length with that of one degree of longitude at the equator.

- (7) ABCD is a square. Equal distances AE, BF, CG, and DH are measured off on

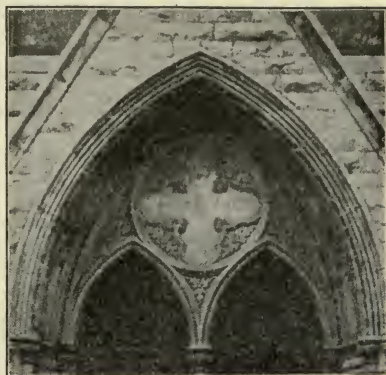
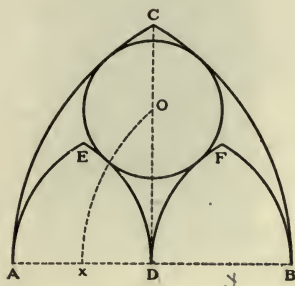


the sides AB, BC, CD, and DA respectively. If the lines AF, BG, CH, and DE are drawn intersecting at Y, Z, W, and X, prove that XYZW is a square. If $AB = a$, and AE is $\frac{1}{3}$ of AB, prove that $AF = \frac{a}{3}\sqrt{10}$, $AX = \frac{a}{10}\sqrt{10}$, $XY = \frac{a}{5}\sqrt{10}$, $FY = \frac{a}{30}\sqrt{10}$; and that the area of XYZW is $\frac{2a^2}{5}$.

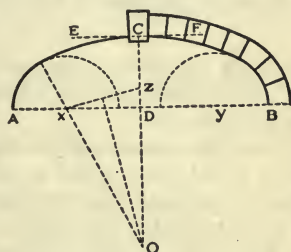
This figure is the basis of an Arabic design used for parquet floors. The solution involves both algebraic and geometric work in concrete setting.

- (8) ABC is an equilateral arch, and CD its altitude. A is the center of the arc BC and B the center of the arc AC. The equilateral arches AED and DFB are erected on AD and BD respectively. D is the center of arc AE and FB, and A and B are centers of arcs ED and DF, each drawn with $\frac{1}{2}AB$ as radius. What is the locus of centers of circles tangent to CA and CB? To ED and DF? To AC and DF? To CB and ED? Construct a circle tangent to the arcs AC, CB, ED, and FD.

This figure is the basis of a common Gothic window design. The solution involves the intersection of loci.

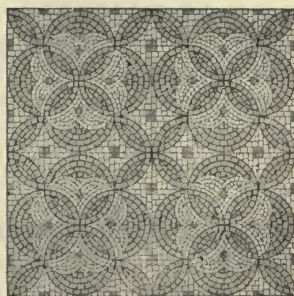
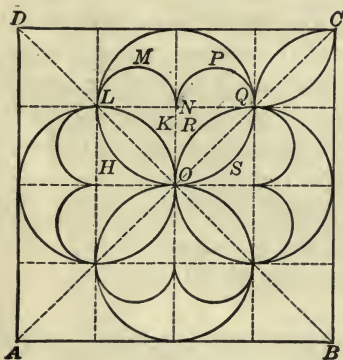


(9) CD is the perpendicular bisector of AB. Equal distances AX and BY are measured off on AD and BD respectively. EF is perpendicular to CD at C. Circles are drawn with X and Y as centers and AX and BY as radii. Construct a circle tangent to EF at C and to circle X. Prove that this circle is also tangent to the circle Y.



If CD is less than $\frac{1}{2}AB$ the part of this figure between lines CD and AB is one form of a three centered arch.

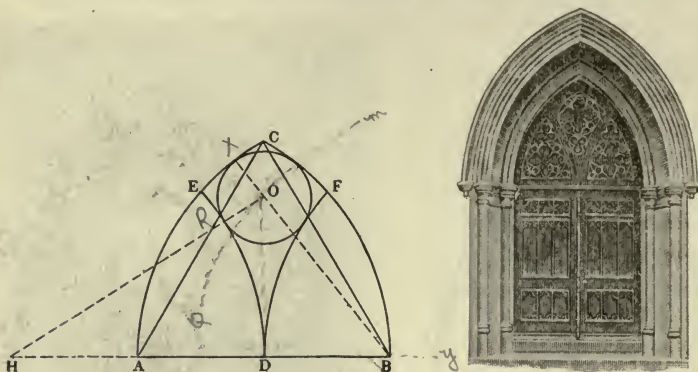
(10) In the drawing below, which is the basis of the mosaic floor design to the right, the circle with center N is inscribed in one of the squares whose side is SH. The arcs



ORQ and OKL are drawn with the vertices of the square as centers and half the side as radius. The semicircles LMN and NPQ are drawn on LN and NQ as diameters. Find the areas of the various figures bounded by circular arcs within this square. Note the symmetry of the whole figure within the square ABCD.

(11) ABC is an equilateral triangle. A and B are centers of arcs BC and AC respectively. CD is the altitude of triangle ABC. Arcs DF and DE, constructed with radii

equal to AB, are tangent to CD at D and intersect AC and CB respectively at E and F. Construct a circle tangent to the arcs DE, DF, AC, and BC.



Suppose the problem solved. Let O be the center of the circle. Connect O with B, the center of arc AC, and with H, the center of arc DE. From triangles ODB and OHB the following equation is derived:

$$(s-r)^2 - (s/2)^2 = (s+r)^2 - s^2,$$

where s is the length of AB and r is radius of the required circle.

This figure is the basis of a church window design. Many problems of this type may be easily obtained.

(12) A quarter-mile running-track has two parallel sides and semicircular ends. Each straightaway section is equal in length to one of the ends. If the track measures exactly one-fourth of a mile at the curb, or inner edge, how much distance does a runner lose in running two feet from the curb? Six feet? What is the area of the track if it is 15 feet wide? What is the area of the inclosed field? What are the dimensions of a rectangular field sufficiently large to contain such a track? What will it cost at \$2.00 per cubic yard to cover such a track with cinders to a depth of 2 inches?—Pettee.

(e) **References to Sources of Problems.**—In connection with the recent search for real applied problems in elementary mathematics numerous bibliographies have been compiled to which reference is here made, as well as to a few other books, aside from current texts, which may be helpful to teachers. Concrete problems should be selected carefully and used wisely. Those which may appear to one class of pupils as real applied problems may seem highly abstract to another class. Probably few problems in the following lists would appear real to all pupils and yet all are likely to find increased interest in any problem which has a concrete origin.

Printed bibliographies.—

(1) A list of 38 titles of books and 21 titles of trade journals, *School Science and Mathematics*, Vol. IX, No. 8, 1909, pp. 788-98.

(2) A more extended list of books on the whole range of applied problems, *School Science and Mathematics*, Vol. VIII, No. 8, November, 1908, pp. 641-44.

From this list Saxelby, Godfrey and Siddons, and Perry may be mentioned especially.

(3) A comprehensive list of books and journals relating to the uses of geometry in architecture, decoration, and design in a new volume entitled *A Source Book of Problems for Geometry*, by Mabel Sykes. Allyn and Bacon, Boston, 1912.

(4) A vast bibliography of suggestive titles, with a classification and discussion of some phases of industrial problems, by a Committee of the National Education Association on the Place of Industries in Public Education, *Proceedings of the Association*, 1910, pp. 652-788.

Collections of problems.—

(1) "Real Problems in Geometry," *Teachers College Record*, March, 1909. A classification and discussion of types of applied problems, by James F. Millis.

(2) "Real Applied Problems in Algebra and Geometry," *School Science and Mathematics*. A collection begun in 1909 by a committee of the Central Association of Science and Mathematics Teachers. The work is still in progress. The problems collected up to November, 1909, have been classified and published in pamphlet form.

Other selected titles.—

(1) *Lessons in Experimental Geometry*, Hall and Stevens. The Macmillan Company, New York, 1905.

(2) *Numerical Problems in Geometry*, J. G. Estill. Longmans, Green and Company, New York, 1908.

(3) *Mensuration*, G. B. Halsted. Ginn and Company, Boston, 1910.

(4) *Elementary Mensuration*, F. H. Stevens. The Macmillan Company, New York, 1908.

(5) *A Notebook of Experimental Mathematics*, Godfrey and Bell. Edward Arnold, London, 1905.

(6) *Elements of Mechanics*, M. Merriman. John Wiley and Sons, New York, 1905.

(7) *Shop Problems in Mathematics*, Breckenridge, Mersereau, and Moore. Ginn and Company, New York, 1910.

(8) *Pocket Companion containing Tables*, etc. Carnegie Steel Company, Pittsburgh, Pa., 1903.

(9) *Leitfaden der Geometrie*, Jahne and Barbisch. Vienna, 1907.

(10) *Raumlehre für Mittelschulen*, Martin and Schmidt. Berlin, 1898.

(11) *Geometrie für die Zwecke des praktischen Lebens*, G. Ehrig. Leipzig, 1906.

(12) *Mathematische Aufgaben*, Schulze and Pahl. Leipzig, 1908.

(13) *Cours abrégée de Géométrie*, Bourlet and Baudoin. Paris, 1907.

(14) *Cahiers d'exécution de dessins géométriques*, M. P. Baudoin. Paris.

(15) *Geometria Intuitiva*, P. Pasquali. Milan.

(16) *Regole di Geometria Pratica*, F. Andreotti. Florence, 1897.

(17) *The Power of Form Applied to Geometrical Tracery*, R. W. Billings. London, 1851.

- (18) *Gothic Architecture in England*, Francis Bond. B. T. Botsford, London, 1905.
- (19) *Les éléments de l'art arabe*, Jules Bourgoïn. Paris, 1879.
- (20) *Pattern Design*, Lewis F. Day. B. T. Botsford, London; Scribner's Sons, New York, 1903.
- (21) *Geometrische Ornamentik*, L. Diefenbach. Max Spielmeier, Berlin.
- (22) *Romano-British Mosaic Pavements*, Thomas Morgan. London, 1886.
- (23) *Decorated Windows, A Series of Illustrations*, Edmund Sharpe. London, 1849.
- (24) *Specimens of Tile Pavements*, Henry Shaw. London, 1858.
- (25) *Specimens of Geometrical Mosaics of the Middle Ages*, Sir Matthew Wyatt. London, 1848.
- (26) *The Teaching of Geometry*, David Eugene Smith. Ginn and Company, Boston, 1911.

PROBLEMS INVOLVING LOCI

(a) **Phraseology.**—While the committee does not wish to prescribe the exact phraseology of any definition, it would recommend greater care in the formulation of the definitions underlying the subject of loci. It is suggested that any definition used should be substantially equivalent to the following:

The locus of a point (or the locus of points) satisfying given conditions is a configuration such that:

- (1) All points lying on the configuration satisfy the conditions;
- (2) All points satisfying the conditions lie on the configuration.

It would seem desirable to make all proofs on loci conform to this definition. It is of course understood that the teacher will lead the pupil up to such a definition thru varied forms of concrete description, such as "path of a point in motion," etc.

(b) **Motion in Geometry.**—It seems well to give some consideration to the place of motion in a well-rounded course in elementary geometry, and to bear in mind that this course is all the geometry to be studied by the majority of high-school pupils. It has recently been urged by prominent European mathematicians that motion should be given a more prominent place at this stage. We may well recall that the space concepts dealt with in our usual courses in geometry are almost entirely to be described as static. There is in theorems and problems on loci a dynamic element that is of importance. The pupil is pretty familiar with motion as a concrete experience, and it seems of first-class importance to idealize some such concrete experiences, until they possess the precision of geometry.

For example, in a given plane we may consider in a way well described as a static configuration the perpendicular bisector of a line-segment joining two points; but when we consider this line as generated by a point moving in the plane in such a way that it is always equidistant from the two given points, we add a dynamic element.

As to phraseology, the expression "locus of points" suggests a static configuration, while the expression "locus of a point" emphasizes the dynamic element and is equivalent in thought to the "path of a point moving with certain prescribed conditions." Both of these phases should have a place in the treatment of loci problems, and thus both forms of expression should be used, the one or the other being more suggestive in different cases. It is even desirable to use different forms in describing a given case to make clear the idea and to cultivate facility in expression.

(c) **Concrete Nature of Loci.**—Contrary to the usual conception, the locus idea is one that may very easily be made concrete and brought down to the comprehension of young pupils. For example, the opening of a book or of a door suggests a variety of loci. The same may be said of many concrete illustrations easily accessible to the pupil.

In this way, loci problems may and should be introduced at certain stages of the subject. For example, in Book I: The locus of a point equidistant from two fixed points, equidistant from two intersecting lines, or from two parallel lines, or at a given distance from a fixed line. In the book on circles, the locus of all points equidistant from a fixed point, the locus of the centers of circles of fixed radius and tangent to a given line, the locus of the centers of all circles tangent to two parallel lines or two intersecting lines, and the locus of the vertices of all triangles having a common base and equal vertex angles. In solid geometry, the locus of points equidistant from a given point, from two given points, from a given plane, from two intersecting planes, from two parallel planes, etc.

(d) **Loci in Problems of Construction.**—Important features of the construction problems in geometry are dependent upon loci considerations which should be emphasized in this connection. For example:

(1) To find the locus in the plane of all points equidistant from three given points, it is necessary to determine the intersection of two loci both of which are straight lines.

(2) To find the locus in the plane of all points equidistant from a fixed point and at a given distance from a given line, it is necessary to find the intersection of two loci one of which is a straight line and the other a circle.

The discussion of the various possibilities in connection with such problems is one of the most valuable exercises for the pupil. For example, as to whether there are one, two, or no points fulfilling the conditions in the second example above. While it may be possible to solve and discuss such problems without using the term locus at all, yet this leads to round-about and awkward explanations, while the language of loci is elegant and concise.

Moreover, facility in the use of this language is not only desirable from the standpoint of the high-school pupil but is of the utmost importance for those who may continue the study of geometry in college.

(e) **To Summarize.**—The locus idea is deserving of a careful and systematic treatment for the following reasons:

(1) It introduces a dynamic element thru the consideration of the idea of motion.

(2) It presents an elegant language for the statement of those propositions on which nearly all of our problems of construction are based.

(3) It aids greatly in the cultivation of space intuition and in emphasizing the important concept of functionality.

(f) Additional Illustrations Appropriate for Use.—

1. Find the locus of all points at a fixed distance from the sides of a triangle, always measuring from the nearest point of a side.

2. Find the locus of points such that the sum of the squares of the distances from two lines intersecting at right angles is 100.

3. Find the locus of the vertices of a regular polygon of a given number of sides that can be circumscribed about a given circle.

4. Find the locus of the midpoints of the sides of regular polygons of a given number of sides that can be inscribed in a given circle.

5. Find the locus of all points from which a given line-segment subtends a given angle.

6. Find the locus of a point the sum of the squares of whose distances from two given points is constant.

7. Find the locus of a point the difference of the squares of whose distances from two given points is constant.

8. Find the locus of all lines drawn thru a given point, parallel to a given plane.

9. Find the locus of a point in space equidistant from three given points not in a straight line.

ALGEBRAIC METHODS IN GEOMETRY

The committee feels that the use of algebraic forms of expression and solution in the geometry courses may well be extended, with advantage to both algebra and geometry, and that this may be done without in any way encroaching upon the field of analytic geometry, which belongs to a later stage of development.

(a) The Notation Should Be More Algebraic.—While it is not feasible or desirable to lay down hard-and-fast rules to standardize the notation of geometry, an examination of current texts makes it evident that some notations in common use are unnecessarily awkward when compared with the notations used in elementary algebra. The notation of geometry is, in general, improved by much use of lower-case letters to represent numerical values, leaving capitals to represent points. This notation is here called algebraic because the student will recognize the relations of equality and inequality much more readily in the familiar notation of algebra than if these relations are presented in a notation not used in algebra.

(b) Algebraic Statement of Propositions.—Many of the theorems of geometry may be stated to advantage in algebraic form, thus giving definiteness and perspicuity and especially emphasizing the notion of functionality. This mode of expression can be made of much value to the student if he is required to translate into English all the symbols involved.

The following are illustrations of the algebraic statement of propositions:

- (1) In any triangle, $a = bh/2$, where a is the area, b is the base and h is the altitude.
- (2) In a right triangle, $c^2 = a^2 + b^2$, where c is the hypotenuse, and a and b are the sides including the right angle.
- (3) In any triangle $c^2 = a^2 + b^2 \pm 2ap$, where a , b , c are sides of the triangle and p is the projection of b on a .
- (4) For any secant and tangent drawn from a point to a circle, we have $t^2 = sx$, where t is the length of the tangent, s is the length of the secant, and x is the length of the external part.

It is not intended to convey the impression that the usual statement of propositions should be replaced by the algebraic statements but rather that the student should be required to translate the one form of statement into the other. The algebraic statements are often superior to the usual statements in point of brevity and conciseness. Moreover, the algebraic statement prepares for the idea of functionality, which is too little understood by persons who are not trained in mathematics beyond the high-school course. That is to say, some appreciation of the influence of changing one part of a configuration on other parts of the configuration can often be gained readily from the algebraic statement.

(c) **Geometrical Construction of Formulas.**—Some propositions can be proved simply and elegantly by methods involving algebra. It is somewhat usual in textbooks on geometry to give a proof of the geometrical statement of such an algebraic formula as $(a+b)^2 = a^2 + b^2 + 2ab$, where a and b are the numerical measures of the line-segments, but to neglect the geometrical construction of the formula. The latter seems to be the point of greatest importance. It is not additional evidence of the validity of the theorem that is sought. That is established in algebra. What is of first-rate importance is to give a geometrical picture of the formula, thus showing a certain geometrical interpretation and to have the student put the result into geometrical phraseology when a and b are line-segments.

The construction of line-segments $a+b$, $a-b$, and of areas ab , $(a+b)^2$, $(a-b)^2$, where a and b are line-segments, should come early in the course. Later, when the requisite theorems are being developed, the further elementary expressions

$$ka, \frac{a}{k}, \frac{ab}{c}, \sqrt{ab}, \sqrt{a^2+b^2}, \sqrt{a^2-b^2}, a\sqrt{k},$$

where a , b , and c are line-segments and k is a positive integer, should be constructed.

This interdependence of algebra and geometry is a matter of no small importance both historically and for subsequent mathematical work. It should be brought out by suitable exercises that the use of algebra often enables one to establish relations from which a geometrical construction can be made readily or to show the nature of a difficulty involved.

For example, to inscribe a square in a semicircle:

If x represents the side of the square and r the radius of the circle,

we have at once from a right triangle that $r^2 = x^2 + x^2/4$ and hence $x = \pm \frac{2}{3}\sqrt{5}r$, which can be constructed from exercises given above.

(d) **Geometric Exercises for Algebraic Solution.**—Some exercises for algebraic solution, such as are found in many recent texts, should find a place in any course in geometry. For example, the following is a suitable exercise after the proposition stating that $a = bh$, where a is the area, b and h are sides of a rectangle:

The area of a rectangle is 480 square inches. Each side of the rectangle is increased 1 inch, and by this change, the area is increased 45 square inches. Find the sides of the rectangle.

Similarly, after the proposition pertaining to secants and tangents to a circle, the following is suitable:

A secant line which passes thru the center of a circle of radius 10 is intersected by a tangent of length 15. Find the length of the external part of the secant.

Such exercises do much to unify geometry and algebra, and may well replace some of the usual exercises.

Finally, after the theorem on the volume of a frustum of a pyramid, a problem like the following has value as an algebraic exercise, altho it is in no sense a real applied problem.

A pier is built of solid concrete construction, in the form of a frustum of a pyramid with square bases. The altitude is twice an edge of the lower base and the area of the lower base is four times that of the upper base. Find the dimensions of each base if the pier contains 600 cubic feet of solid concrete.

SECTION E. SYLLABUS OF GEOMETRY

PREFACE TO LISTS OF THEOREMS

(1) *Lists not exhaustive.*—The lists of theorems which follow are not to be taken as exhaustive, and it is distinctly understood that theorems may be added at the discretion of the teacher. For example, the theorem on the existence of regular polyhedra may find a place in certain courses. Some theorems are omitted only with the understanding that they may be inserted as exercises for the student; some such possible exercises are:

In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

The medians of a triangle meet in one point which divides each median in the ratio 1:2.

To divide a given straight line-segment in extreme and mean ratio.

To find the area of a triangle in terms of its sides.

To construct a square having a given ratio to a given square.

The surface of a sphere is equivalent to the area of four great circles.

(2) *Logical order.*—Altho there is some indication of a possible order in the lists, there is no intention of specifying any definite order. It would be impossible to carry out as a whole precisely the order stated below.

In several connections the words "corollary to" or "synonymous to"

may seem to imply an order. These phrases are used only to indicate the reason for putting the theorem quoted in the group in which it appears. Thus in Group II on p. 41, it would not be clear in every case that each theorem is a corollary of plane geometry without such a suggestion of possible derivation.

It should be noticed that some logical arrangements would necessitate the insertion of the theorems omitted in this list. Such an insertion is entirely in the spirit of this report, as is also any conceivable change in the order, except where specified explicitly in the report.

(3) *Subsidiary theorems.*—A number of theorems omitted in the lists below may well be given as ordinary statements in the course of the text as corollaries, or as remarks, without the emphasis which attaches to formal theorems. Among such general statements which should by all means be made at the proper points are the following:

No triangle can have more than one right angle or more than one obtuse angle.

The third angle of a triangle can be found if two are known.

An equilateral triangle is equiangular.

The square on a side of a right triangle adjacent to the right angle is equal to the square on the hypotenuse minus the square on the other side.

Thru three points not in a straight line not more than one plane can be passed.

The areas of two spheres are to each other as the squares of their radii; their volumes as the cubes of their radii. (Like statements for other solids.)

The number of such statements is exceedingly large and all of them could not be given in any syllabus. A large majority are at the present time stated, if at all, in the course of the reading-matter, or in exercises, and not as explicit theorems. It is understood, and indeed expected, that these statements, together with many which are omitted from the lists of theorems below, should be treated in this manner.

(4) *Informal proofs.*—The theorems given below under the heading "Theorems for Informal Proofs" should be stated at the proper points in the text and in theorem form, or as postulates. Their proofs, however, can well be omitted where this omission is suggested, or be made exceedingly informal by the insertion of a single phrase which will give the proper suggestion for the proof. Many other theorems which are equally obvious are not stated because they occur more naturally as corollaries or as exercises. (See the preceding paragraph.)

Regarding the method of proof in general, while the demonstrations should remain as logical as they are at present, it is suggested that the *formalities* of logic, as such, be frequently dispensed with to a very considerable extent and that the propositions be frequently stated and proved in language resembling that to be found in any other mathematical textbook. This is, indeed, the style of many classical treatises, such as Legendre's or Euclid's. It is certainly satisfactory and there is no reason why the proof should not remain quite as logical when the older style is followed.

The symbolic form of demonstration which appears in many texts should be regarded simply as a shorthand expression of a complete proof in ordinary English phraseology. The latter should be given by the student in all cases. The ability to pass from the symbolic form to ordinary English, that is, to translate the shorthand into the language of everyday life, should be constantly tested by the teacher, for the same reason that the formulas of algebra derive their real meaning and power from the thought content which the student can attach to them.

(5) *Arrangement for emphasis.*—The main list of theorems is divided into several heads, each group being introduced by a theorem of suitable importance upon which the rest of the theorems in that group depend more or less closely. This arrangement has been selected in order to emphasize the importance of a few major propositions, namely, those which carry a maximum of applications and from which the rest can be derived, thus serving as a nucleus for the whole of geometry.

This effort to gain emphasis has been carried out still farther by printing the theorems in different grades of type so that those of fundamental importance and of basal character are printed in black-face type; those of considerable importance which are secondary only to the preceding ones are printed in italics. A number of other theorems are printed in roman type, while the least important are printed in small type. The latter (small-type theorems) may be omitted without serious danger, or they may be used as corollaries or exercises instead of receiving the emphasis which attaches to a theorem; in fact, probably no injury would result from a similar treatment of many of the theorems stated in roman type.

The distinction in emphasis is desirable not only for guidance in omitting theorems in courses which are necessarily abbreviated, but it is also of the highest importance in courses in which all of the theorems are given. An orderly classification of theorems in the student's mind, a notion of the dependence of the minor theorems on the more basal ones, and an appreciation of their relative importance are of the utmost direct value to the student and furnish him with the only possible means of permanently retaining geometrical knowledge in usable form. The direct value mentioned arises both from the power acquired and also from the essential grasp of the subject, which is the purpose of education. It is a fundamental characteristic of the mind from which there is no escape that any clear impression of a vast field must have exactly such distinctions in emphasis as are outlined here for geometry. These statements and this arrangement are intended to be of assistance to the teacher in guiding him as to the emphasis to be laid upon theorems during the course and especially at the completion of a given book or chapter.

(6) *Trigonometric ratios.*—Attention is called to the paragraphs under XIII, 2-4, on the computation of two-place tables of *sines*, *cosines*, and *tangents* from actual measurements, provided the pressure of time due to

examining bodies is not too great. This work can be done with about the same amount of effort that is expended by the student on the ordinary geometrical theorems of the same class. Its importance is due to the fact that such a small table will really present the fundamental ideas of trigonometry and will enable the student to solve right triangles in the trigonometric sense.

(7) *Abbreviations*.—In a large number of instances theorems are stated in condensed or abbreviated form and the statement of a number of theorems is often combined into one. This is done only for the purpose of reducing the length of this report. It is to be understood that such abbreviated statements are made only for the teacher and should not be presented to the student in this form. In particular, it is probably preferable to use words instead of letters in statements for high-school pupils of such theorems as those in II, 1-4.

The numbers which follow each of the theorems are references to a syllabus prepared by a committee of the Association of Mathematics Teachers of New England, 1906.

(8) *Omissions*.—Since the quantities which appear in geometry are often treated by means of their numerical measures, the introduction of any useful algebraic facts is completely justifiable; in particular, the algebraic *theory of proportion* may well be employed; but such algebraic material is not included in this syllabus since the syllabus deals primarily with geometric topics.

The following list shows the omissions from the New England Syllabus:

Plane geometry omissions: C₂, G₁₂ (2d part), G₂₀, J₁₃, L₂, N₆ (2d part), N₁₃, P₁, P₂ (see note to XV, 1), P₆, P₁₂, T₂, T₄, T₅.

Solid geometry omissions: E₇, E₈, E_{9b}, F₁₂, F₁₃, F₁₄, H₂ (but see IV, 7), K₂, K₃ (see note to I, 2), M₆, M₇, M₈, M₉, Q₃, Q₄, Q_{8b}, Q₉ (see VI, note), R₁, R₂ (see note to VI, 13), R₉, R₁₀, R₁₃, R₁₄ (see note to VI, 22), S₃ (see preface, 3), S₇.

THEOREMS OF PLANE GEOMETRY

I. Theorems for Informal Proof

(The following theorems may be stated as assumptions, or may be given such informal proof as the circumstances may demand.)

1. All straight angles are equal. [*]
2. All right angles are equal. [*]
3. The sum of two adjacent angles whose exterior sides lie in the same straight line equals a straight angle. [J₁.]
4. If the sum of two adjacent angles equals a straight angle their exterior sides form a straight line. [J₂.]

* Reference numbers are to the New England Syllabus. Where an asterisk [*] replaces the reference number, the theorem is not contained in that syllabus.

5. Only one perpendicular can be erected from a given point in a given line. [G3.]

6. The length of a circle (circumference) lies between the lengths of perimeters of the inscribed and circumscribed convex polygons. [P13.]

(It is recommended that this statement be used as a definition to be inserted at context.)

7. The area of a circle lies between the areas of inscribed and circumscribed convex polygons. [P14.]

(It is recommended that this statement be used as a definition to be inserted at context.)

8. Two lines parallel to the same line are parallel to each other. [*]

9. Vertical angles are equal. [J3.]

(Very informal proof sufficient.)

10. Complements of equal angles are equal. [*]

11. Supplements of equal angles are equal. [*]

12. The bisectors of vertical angles lie in a straight line. [J4.]

13. Any side of a triangle is less than the sum of the other two and greater than their difference. [*]

14. A diameter bisects a circle. [A5.]

15. A straight line intersects a circle at most in two points. [G6.]

II. Congruence of Triangles

1. Any two triangles¹ ABC and A'B'C' are congruent if:

(1) $a = a'$ $b = b'$ $C = C'$ [A1.]

(2) $a = a'$ $B = B'$ $C = C'$ [A2.]

(3) $a = a'$ $b = b'$ $c = c'$ [A3.]

(4) $a = a'$ $c = c'$ $C = C' = 90^\circ$ [A4.]

(State these in detail and in English. See preface, article 7.)

2. A triangle is determined when the following are given: (1) a, b, C ;

(2) a, B, C ; (3) a, b, c ; (4) $a, c, C = 90^\circ$. [*]

(Synonymous to 1.)

3. Construction of triangles from given parts; measurement of unknown parts by ruler and protractor. Given: (1) a, b, C ; (2) a, B, C ; (3) a, b, c ; (4) a, c, C , possibly two solutions. [*]

(This is the fundamental elementary idea of trigonometry.)

4. In any two triangles if $a = a'$ and $b = b'$, either of the inequalities $c > c'$ or $C > C'$ is a consequence of the other. [O3, O4.]

III. Congruent Right Triangles

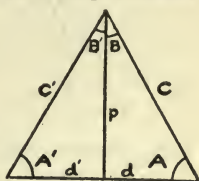
1. Two right triangles are congruent if, aside from the right angles, any two parts, not both angles, in the one are equal to corresponding parts of the other. [A4.]

(Very important subcase of II, 1.)

¹In this syllabus the angles of a triangle ABC are denoted by the capital letters A, B, and C; the sides are denoted by small letters a, b , and c , where a is the side opposite the angle A, etc.

2. If two oblique lines c and c' be drawn from a point in a perpendicular p to a line AA' , cutting off distances d and d' , then any one of the equalities $c=c'$, $d=d'$, $A=A'$, $B=B'$, is a consequence of any other. [G5.]

3. A diameter perpendicular to a chord bisects the chord, the subtended angle at the center, and the subtended arc; conversely, a diameter which bisects a chord is perpendicular to it. [G5b, G8.]



(Corollary to 2. See also IV, 3.)

4. If two oblique lines c and c' be drawn from a point in a perpendicular p to a line AA' , cutting off unequal distances d and d' , then either of the inequalities $c > c'$, $d > d'$, is a consequence of the other. [O5, O6.]

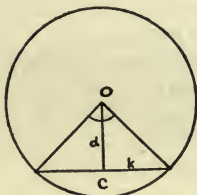
(In particular, c is greater than p .)

5. If, in a triangle ABC , $a=b$, the perpendicular from C on c divides the triangle into two congruent triangles. [*]

6. In a triangle ABC , either of the equations $a=b$, $A=B$, is a consequence of the other. [G1, G2.]

7. In a triangle ABC , either of the statements $a > b$, $A > B$, is a consequence of the other. [O1, O2.]

IV. Subtended Arcs, Angles, and Chords



1. In the same circle, or in equal circles, any one of the equations $d=d'$, $k=k'$, $c=c'$, $O=O'$, is a consequence of any other one of them. [A6, 7, 8, 9, G9.]

2. Any one of the inequalities (see figure)

$$d < d', O > O', c > c', k > k'$$

is a consequence of any other one of them. [O7, 8.]

3. In any circle an angle at the center is measured by its intercepted arc. [J8.]

(Only the commensurable case.)

4. If a circle is divided into equal arcs, the chords of these arcs form a regular polygon. [G12, last part.]

5. To construct an angle equal to a given angle. [J14.]

(Regular polygons may be constructed approximately by means of a protractor. In the same way other approximate constructions may be introduced which depend upon the protractor.)

V. Perpendicular Bisectors

1. The perpendicular bisector of a line-segment is the locus of points equidistant from the ends of the segment. [S1.]

2. To draw the perpendicular bisector of a given line-segment. [G14.]

3. To erect a perpendicular at a given point in a line. [*]

(Corollary to 2.)

4. To drop a perpendicular from a given point to a given line. [D5.]
(Corollary to 2.)
5. To bisect a given arc or angle. [G15, 16.]
(See III, 3.)
6. To inscribe a square in a circle. [G18.]
7. *One and only one circle can be circumscribed about any triangle.* [G13.]
8. Three points determine a circle. Two circles can intersect, at most; in two points; this will happen when the distance between their centers is less than the sum of the radii and greater than the difference of the radii. [G7.]
(Corollary to 7.)
9. Given an arc of a circle, to find its center. [*]
(Corollary to 7.)
10. A circle may be circumscribed about any regular polygon. [G13, third part.]
11. The perpendicular bisectors of the sides of a triangle meet in a point. [T3.]

VI. Bisectors of Angles

1. The bisector of any angle is the locus of points equidistant from the sides of the angle. [S2.]
2. *A circle can be inscribed in any triangle.* [G13, second part.]
(Construction to be given.)
3. A circle can be inscribed in any regular polygon. [G13, last part.]
4. Of the inscribed and circumscribed regular polygons of n and $2n$ sides for a given circle, to draw the remaining three polygons when one is given. [G17.]
5. The bisectors of the angles of any triangle meet in a point. [T1.]
(Corollary to 2.)

VII. Parallels

1. When two lines are cut by a transversal the alternate interior angles are equal if, and only if, those two lines are parallel. [Half of D1, 2.]
When two lines are cut by a transversal, the alternate interior angles are unequal if, and only if, the lines are not parallel.
(Synonymous to 1.)
2. When two lines are cut by a transversal the corresponding angles are equal, and the two interior angles on the same side of the transversal are supplementary if, and only if, the two lines are parallel. [Half of D1, 2.]
(Corollary to 1.)
3. Two lines in the same plane perpendicular to the same line are parallel. [D4, G4.]
(Only one perpendicular can be let fall from a point without a line to that line.
Synonymous to 3.)
4. A line perpendicular to one of two parallels is perpendicular to the other also. [D3.]
(Corollary to 1.)

5. *If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.* [J7.]

6. Thru a given point to draw a straight line parallel to a given straight line. [D6.]

7. *A parallelogram is divided into two congruent triangles by either diagonal.* [*]

8. In any parallelogram the opposite sides are equal, the opposite angles are equal, the diagonals bisect each other. [D7.]

(Corollary to 7.)

9. In any convex quadrilateral, if the opposite sides are equal, or if the opposite angles are equal, or if one pair of opposite sides are equal and parallel, or if the diagonals bisect each other, the figure is a parallelogram. [D8.]

VIII. Angles of a Triangle

1. In any triangle the sum of the angles is two right angles. [J5(b).]

2. In any triangle, any exterior angle is equal to the sum of the two opposite interior angles. [J5(a).]

(Synonymous to 1.)

3. The sum of the interior angles of a polygon of n sides is $2(n-2)$ right angles. [J6.]

4. To inscribe a regular hexagon in a circle. [G19.]

(To construct an angle of 60° . Synonymous to 4.)

IX. Inscribed Angles

1. An angle inscribed in a circle is measured by half of its intercepted arc. [J9.]

2. *Angles inscribed in the same segment are equal to each other.* [*]

3. An angle inscribed in a semicircle is a right angle. [*]

4. The two arcs intercepted by parallel secants are equal. [G11.]

5. The angle between a tangent and a chord is measured by half the intercepted arc. [J10.]

6. The angle between any two lines is measured by half the sum, or half the difference, of the two arcs which they intercept on any circle, according as their point of intersection lies inside of, or outside of, the circle. [J11, 12.]

7. *The tangent to a circle at a given point is perpendicular to the radius at that point.* [L1, 3.]

8. For a given chord, to construct a segment of a circle in which a given angle can be inscribed. [J15.]

9. To draw a tangent to a given circle thru a given point. [L4.]

10. The tangents to a circle from an external point are equal. [G10.]

(Corollary to 7.)

X. Segments Made by Parallels

1. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any other transversal. [D9.]
2. The segments cut off on two transversals by a series of parallels are proportional. [See N10.]
(Only the commensurable case.)
3. A line divides two sides of a triangle proportionally, the segments of the two sides being taken in the same order, if, and only if, it is parallel to the third side. [N1, 2.]
(Only the commensurable case.)
4. To divide a line-segment into n equal parts or into parts proportional to any given segments. [N9, 10.]
5. To find a fourth proportional to three given line-segments. [N11.]

XI. Similar Triangles

1. Two triangles ABC and $A'B'C'$ are similar if

(1)	$A=A'$	$B=B'$	$C=C'$	[N3.]
or (2)	$a=ka'$	$b=kb'$	$C=C'$	[N4.]
or (3)	$a=ka'$	$b=kb'$	$c=kc'$	[N5.]

where k is a constant factor of proportionality.

(See preface, article 7.)

2. Given a fixed point P and a circle C , the product of the two distances measured along any straight line thru P , from P to the points of intersection with C , is constant. This product is also equal to the square of the tangent from P to C if P is an external point. [N18.]

3. The bisector of any angle of a triangle divides the opposite side into segments proportional to the adjacent sides. [Half of N6.]

4. To construct a triangle similar to a given triangle. [*]

(Drawing triangles to scale; measurements of remaining parts to scale. Basal in trigonometry.)

XII. Similar Figures

1. Polygons are similar if, and only if, they can be decomposed into triangles which are similar and similarly placed. [N7, 8.]

2. Regular polygons of the same number of sides are similar. [N14.]

3. The perimeters of similar polygons are proportional to any two corresponding lines of the polygons. [N15.]

4. The circumferences of any two circles are proportional to their diameters, thus $c=2\pi r$, where π is constant. [P15.]

($\pi=3.14\dots$ to be computed later.)

5. To construct a polygon similar to a given polygon. [*]

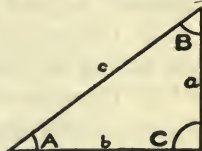
(Drawings to scale; maps, house plans; readings from drawings; plotting of measurements. Essential in surveying.)

XIII. Similar Right Triangles

(The committee feels that numbers 2, 3, 4 following should have a place where time for their discussion can be secured, which will doubtless be the case except under pressure from examining bodies.)

1. Any two right triangles are similar if an acute angle of the one is equal to an acute angle of the other, or if any two sides of one are proportional to the corresponding sides of the other. [*]

2. For a given acute angle A , the sides of any right triangle ABC ($C=90^\circ$) form fixed ratios, called the sine (a/c), the cosine (b/c), the tangent (a/b). [*]



3. Computation of a two-place table of sines, cosines, tangents from actual measurements. [*]

(Probably a two-place table for every 5° or 10° ; to be done by students, preferably on squared paper.)

4. Solution of right triangles with given parts by use of the preceding table of ratios. [*]

(Height and distance exercises.)

XIV. Right Triangles

1. In any right triangle ABC the perpendicular let fall from the right angle upon the hypotenuse divides the triangle into similar right triangles, each similar to the original triangle. [*]



2. The length of the perpendicular p is the mean proportional between the segments m and n of the hypotenuse; i.e., $p^2=mn$. [P8.]

3. Either side, a or b , is the mean proportional between the whole hypotenuse c and the adjacent segment m or n ; that is, $a^2=cm$; $b^2=cn$. [P9.]

4. To find a mean proportional between two given line-segments. [N12.]

5. The sum of the squares of the two sides of a right triangle is equal to the square of the hypotenuse; $a^2+b^2=c^2$. [P10.]

(It should be noticed that the proposition can be proved either algebraically or geometrically.)

6. In any triangle ABC , if B is less than 90° , then $b^2=a^2+c^2-2cm$; if B is greater than 90° , then $b^2=a^2+c^2+2cm$, where m is the projection of a on c . [P17.]

(See figure under 2.)

7. Given the radius of a circle and a perimeter of an inscribed regular polygon of n sides, to find the perimeter of the circumscribed regular polygon of n sides and the perimeter of the inscribed regular polygon of $2n$ sides.

[G17. See also X4.]

8. To calculate π approximately. [*]

XV. Areas

1. The area of a rectangle is the product of its base and its altitude; i.e., $a=bh$. [P1, 2, 3.]

(This formula may be taken as the definition of area.)

2. *Parallelograms or triangles of equal bases and altitudes are equivalent.* [C1.]

3. The area of a parallelogram is the product of its base and its altitude; i.e., $a = bh$. [P4.]

4. The area of a triangle is one-half the product of its base and its altitude; i.e., $a = \frac{1}{2}bh$. [P5.]

5. The area of a trapezoid is one-half the product of its altitude and the sum of its bases; i.e., $a = \frac{1}{2}(b_1 + b_2)h$. [P7.]

6. The areas of similar triangles or polygons are proportional to the squares of corresponding lines. [N16, 17.]

7. The area of a regular polygon is one-half the product of its perimeter and its apothem. [P11.]

8. The area of any circle is one-half the product of its circumference and its radius; i.e., $a = \pi r^2$. [P16.]

9. The areas of two circles are proportional to the squares of their radii. [*]
(May be treated as suggested in preface, article 3.)

10. To construct a square equivalent to the sum of two given squares. [*]
(Pythagorean proposition.)

11. To construct a square equivalent to a given rectangle. [C3.]
(Mean proportional. See X6.)

THEOREMS OF SOLID GEOMETRY

In this part the same general principles apply as were stated in the preface above.

Thruout, but particularly in divisions I and II below, very great emphasis should be laid upon the student's real grasp of the conceptions, of the space figures, and of the significance of the theorems. While the theorems in division I will be seen to need little or no suggestion of proof, it is a mistake to suppose that they can be hastened over; on the contrary, even in these, the teacher should spare no pains to make sure that the student's mental picture is quite vivid, resorting to formal proof when necessary. To this end, illustrations, figures, models, various forms of presentation, and all such aids are legitimate thruout the course in solid geometry.

I. Theorems for Informal Proof

1. If two planes cut each other, their intersection is a straight line. [S4.]

2. Two dihedral angles have the same ratio as their plane angles. [K2, 3, 4.]

(Equivalent to K3.)

3. Every section of a cone made by a plane passing thru its vertex is a triangle. [M4.]

4. Every section of a cylinder made by a plane passing thru an element is a parallelogram. [M2.]

5. The area of a sphere lies between the areas of circumscribed and inscribed convex polyhedrons. [*]

(It is recommended that this statement be used as a definition to be inserted at context.)

6. The volume of a sphere lies between the volumes of circumscribed and inscribed convex polyhedrons. [*]

(It is recommended that this statement be used as a definition to be inserted at context.)

7. The projection of a straight line upon a plane is a straight line. [S8.]

II. Corollaries from Plane Geometry

(The ability to make the transfer from plane geometry to solid geometry, and vice versa, in forming conceptions and in logical deductions is of the utmost importance. The following theorems are easily reducible to plane geometry in at most two or three planes. The intention is that careful proofs be given, but the student should see that these theorems result immediately from known theorems of plane geometry.)

1. The intersections of two parallel planes with any third plane are parallel. [F1.]

2. A plane containing one and only one of two parallel lines is parallel to the other. [F7.]

3. If a straight line is parallel to a plane, the intersection of the plane with any plane drawn thru the line is parallel to the line. [*]

4. Thru a given point only one plane can be passed parallel to two straight lines not in the same plane. [F10.]

(Derived from 2.)

5. Thru a given straight line only one plane can be passed parallel to any other given straight line in space, not parallel to the first. [F11.]

(Derived from 2.)

6. Thru a given point only one plane can be drawn parallel to a given plane. [F9.]

(Synonymous to 4.)

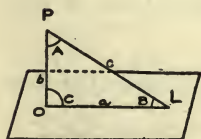
7. If a perpendicular PO be let fall from a point P to a plane L, any one of the equalities

$$a = a', c = c', B = B', A = A'$$

is a consequence of any other of them; and any one of the inequalities

$$a > a', c > c', B < B', A > A'$$

is a consequence of any other one of them. [O10, H7. See also S8.]



8. The perpendicular PO is shorter than any oblique line. [O9.]

9. Two straight lines are parallel to each other if, and only if, they are both perpendicular to some one plane. [F2, 3.]

10. If two straight lines are parallel to a third, they are parallel to each other. [F4.]

(Derived from 9.)

11. Two planes are parallel to each other if, and only if, they are both perpendicular to some one straight line. [F5, 6.]

(Derived from 9.)

12. The locus of points equidistant from the extremities of a straight line is a plane perpendicular to that line at its middle point. [S5.]

13. If two straight lines are cut by three parallel planes, their corresponding segments are proportional. [See M1.]

14. The locus of points equidistant from two intersecting planes is the figure formed by the bisecting planes of their dihedral angles. [S6.]

III. Planes and Lines

1. If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to every line in their plane thru the foot of the perpendicular. [E1.]

2. Every perpendicular that can be drawn to a straight line at a given point lies in a plane perpendicular to the line at the given point. [E2.]

(Corollary to 1.)

3. Thru any point only one plane can be drawn perpendicular to a given line. [E5.]

(Corollary to 1, and II, 11.)

4. Thru a given point only one perpendicular can be drawn to a given plane. [E6.]

(Corollary to 1.)

5. If two angles have their sides respectively parallel and lying in the same direction, they are equal, and their planes are parallel. [K1, F8.]

6. *If a line meets its projection on a plane, any line of the plane perpendicular to one of them at their intersection is perpendicular to the other also.* [*]

7. Between any two straight lines not in the same plane, one and only one common perpendicular can be drawn, and this common perpendicular is the shortest line that can be drawn between the two lines. [E12.]

8. *Two planes are perpendicular to each other if, and only if, a line perpendicular to one of them at a point in their intersection lies in the other.* [E3, 4.]

9. If a straight line is perpendicular to a plane, every plane passed thru the line is perpendicular to the first plane. [E9a.]

(Corollary to 8.)

10. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane. [E10.]

(Corollary to 8.)

11. *Thru a given straight line oblique to a plane, one and only one plane can be passed perpendicular to the given plane.* [E11.]

12. The acute angle which a straight line makes with its own projection on a plane is the least angle which it makes with any line of the plane. [O13.]

13. Two right prisms are congruent if they have congruent bases and equal altitudes. [B1.]

14. *If parallel planes cut all the lateral edges of a pyramid, or a prism, the sections are similar polygons; in a prism, the sections are congruent; in a*

pyramid, their areas are proportional to the squares of their distances from the vertex. [M1.]

(See II, 13.)

15. Every section of a circular cone made by a plane parallel to its base is a circle, the center of which is the intersection of the plane with the axis. [M5.]

16. Parallel sections of a cylindrical surface are congruent. [M3.]

IV. Spheres

1. Every section of a sphere made by a plane is a circle. [M10.]

(Several corollaries may be added.)

2. The intersection of two spheres is a circle whose axis is the line of centers. [H4.]

3. The shortest path on a sphere between any two points on it is the minor arc of the great circle which joins them. [O14.]

4. A plane is tangent to a sphere if, and only if, it is perpendicular to a radius at its extremity. [M11, 12, 13.]

5. A straight line tangent to a circle of a sphere lies in a plane tangent to the sphere at the point of contact. [*]

6. The distances of all points of a circle on a sphere from either of its poles are equal. [H1.]

7. A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is the pole of the great circle passing thru these points. [H3.]

8. A sphere can be inscribed in or circumscribed about any given tetrahedron. [H5.]

9. A spherical angle is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary). [K5.]

V. Spherical Triangles and Polygons

(Every theorem stated here may also be stated as a theorem on polyhedral angles.)

1. Each side of a spherical triangle is less than the sum of the other two sides. [O11 (b). See also (a).]

2. The sum of the sides of a spherical polygon is less than 360° . [O12 (b). See also (a).]

3. The sum of the angles of a spherical triangle is greater than 180° and less than 540° . [K8.]

4. If $A'B'C'$ is the polar triangle of ABC , then, reciprocally, ABC is the polar of $A'B'C'$. [K6.]

5. In two polar triangles each angle of the one is the supplement of the opposite side in the other. [K7.]

6. Vertical spherical triangles are symmetrical and equivalent. [C8, H6.]

7. Two triangles² on the same sphere are either congruent or symmetrical if

$a = a'$	$b = b'$	$c = c'$	[B2 (b). See also (a).]
or $a = a'$	$b = b'$	$C = C'$	[B3 (b). See also (a).]
or $a = a'$	$B = B'$	$C = C'$	[B4 (b). See also (a).]
or $A = A'$	$B = B'$	$C = C'$	[B5 (b). See also (a).]

8. Either of the equations $a = b$, $A = B$ is a consequence of the other. [*]

² The same notation is used as in the plane triangles.

VI. Mensuration

(The relation between the areas and volumes of similar solids may be treated as corollaries in individual cases. See preface, article 3. It is understood that certain statements concerning limits may be assumed either explicitly or implicitly; these are not stated as theorems. See Q3, 4, 9, R9, 10.)

1. An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism. [C4.]

2. A plane passed thru two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms. [C6.]

3. The lateral area of a prism is the product of a lateral edge and the perimeter of a right section. [Q1.]

(Corollary of plane geometry.)

4. The lateral area of a regular pyramid is one-half the product of the slant height and the perimeter of the base. [Q2.]

(Corollary of plane geometry.)

5. The lateral area of a right circular cylinder is the product of the altitude and the circumference of the base; i.e., $s = 2\pi rh$. [Q5.]

6. The lateral area of a right circular cone is one-half the product of the slant height and the circumference of the base; i.e., $s = \pi rl$. [Q6.]

7. The lateral area of a frustum of a regular pyramid is one-half the product of the slant height and the sum of the perimeters of the bases. [Q7.]

8. The lateral area of a frustum of a right circular cone is one-half the product of the slant height and the sum of the circumferences of the bases. [Q8 (a).]

9. The area of a zone is the product of its altitude and the circumference of a great circle; i.e., $s = 2\pi rh$. [Q10.]

(Lemma for 10 below.)

10. The area of a sphere is the product of its diameter and the circumference of a great circle; i.e., $s = 4\pi r^2$. [Q11.]

11. The area of a lune is to the surface of a sphere as the angle of the lune is to 360° . [Q12.]

12. The area of a spherical triangle is to the area of the sphere as its spherical excess is to 720° . [Q13.]

13. The volume of a rectangular parallelopiped is the product of its three dimensions. [R1, 2, 3.]

(This may be taken as a definition.)

14. The volume of any parallelopiped is the product of its base and altitude. [C5, R4.]

15. The volume of any prism is the product of its base and its altitude. [R5, 6.]

16. The volume of any pyramid is one-third the product of its base and its altitude. [C7, R7, 8.]

17. The volume of a circular cylinder is the product of its base and its altitude, i.e., $v = \pi r^2 h$. [R11.]

18. The volume of a circular cone is one-third the product of its base and its altitude; i.e., $v = \frac{1}{3}\pi r^2 h$. [R12.]

19. The volume of a spherical sector is one-third the product of the radius and the zone which is its base: i.e., $v = \frac{2}{3}\pi r^2 h$. [R15.]

20. The volume of a sphere is one-third the product of its radius and its area; i.e., $v = \frac{4}{3}\pi r^3$. [R16.]

(The wording suggests a proof, but that proof is by no means prescribed. The wording is convenient, the proof may even preferably follow 21 below.)

The following two theorems, while not thought by the committee to be indispensable, offer for both student and teacher an outlook for that larger view of geometry and of mathematics as a whole which is very desirable. They forecast important principles in future mathematical courses; they are capable of the most practical direct applications; they offer a possibility of organizing and retaining the important mensuration formulas given above.

21. *If two solids contained between the same parallel planes are such that their sections by a plane parallel to those planes are equal in area, the two solids have the same volume.* [*]

("Cavalieri's Theorem." Formal proof should not be given.)

22. The volume of any sphere, cone, cylinder, pyramid, or prism, or of any frustum of one of these solids intercepted by two parallel planes, is given by the formula $v = \frac{1}{6}h(t + 4m + b)$, where t is the area of the upper base, b that of the lower base, m that of a base midway between the two, and where h is the perpendicular distance between the two parallel planes. [See R13, 14.]

(This formula also applies to any so-called prismatoid; it is conveniently useful in practical affairs. It should not be proved for the general case, but each separate solid mentioned above, numbers 1 to 20, can be shown to conform to this rule, by a direct check.)

MAXIMUM AND MINIMUM LISTS

The committee feels that it would be impossible to set up a minimum list, or a maximum list, of theorems for college-entrance examinations or for other broad purposes, which would meet with any widespread approval; and that the influence of such a list, if it were generally accepted, would be pernicious in leading to a total disregard of many minor facts of geometry which deserve at least passing notice.

Recognizing, however, a practical demand for some criterion on the part of the college examiner, as well as on the part of the teacher, the committee makes the following recommendations for the guidance of examiners and teachers:

(a) All theorems in black-face type in this syllabus should be thoroly known. The student should be able to state each of these upon a clear suggestion of its topic; and to demonstrate each of them without hesitation in accordance with some definite logical order. He should know why they are important: in particular, he should be ready to mention other theorems in geometry closely connected with them and he should know any important concrete applications of these theorems in ordinary life.

(b) All theorems in italics should be known to the student substantially in the form here given, but some latitude may be allowed in combining

parts of these with parts of other theorems. The student should be able to prove each of these theorems on fairly short notice and he should have a reasonable idea of the importance of each of them.

(c) The theorems printed in ordinary roman type should be familiar to the student when they are stated by the examiner, and the student should be able to make a proof for any one of them if allowed a reasonable interval for thought.

(d) The theorems printed in small type, and indeed many other facts of geometry not given in the syllabus, may be used by the examiner with the understanding that they are to be regarded in examinations as of the nature of exercises rather than as theorems with which the student is supposed already to have considerable familiarity.

(e) However, the committee would suggest (1) that examination questions involving the trigonometric ratios, XIII, 2, 3, 4, p. 39, be accompanied by *alternative* questions on other topics, since some schools may not find time for these applications; and (2) that no questions be given by examiners involving proofs of theorems which may properly be taken as assumptions or as definitions, such, for instance, as I, 6, 7, p. 34; XV, 1, p. 39; VI, 13, 21, pp. 44, 45.

CONCLUSION

It should be said that the members of the committee are not entirely agreed as to certain minor details of this report. For example, some would place among the exercises certain propositions now in small type; others would prefer to put some theorems in black-face type which are now in italics; others would prefer three types of propositions instead of four; and some would modify certain postulates and would consider as postulates or as propositions to be demonstrated certain theorems included in the list of those requiring only informal proof. The committee does not regard these minor matters of any great consequence, and therefore wishes to be considered as approving the spirit and general tenor of the report, rather than as giving individual sanction to all such details.

HERBERT E. SLAUGHT, *Chairman*

The University of Chicago, Chicago, Ill.

WILLIAM BETZ

East High School, Rochester, N.Y.

EDWARD L. BROWN

North High School, Denver, Colo.

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Teachers College, Columbia University,
New York City, N.Y.

EUGENE R. SMITH

The Park School, Baltimore, Md.

MABEL SYKES

Bowen High School, Chicago, Ill.

It is desired that this Final Report may have a wide circulation among teachers of geometry and mathematicians in general.

Bound copies of the report may be secured *gratis* upon application to the Commissioner of Education, Department of the Interior, Washington, D.C.

COMMENTS ON THE REPORT

EARLE R. HEDRICK, PROFESSOR OF MATHEMATICS, UNIVERSITY OF MISSOURI,
COLUMBIA, MO.

The Committee of Fifteen, whose report I have the honor to present for your consideration, has had a history which is perhaps unique. The committee was appointed formally at the meeting of the Association at Denver in 1909; its labors have been practically continuous since that time; its membership has remained intact; the report itself has received the widest discussion and distribution, not alone among the members of the committee, but rather thruout the entire membership of this great Association and the other societies which have interested themselves in it. It now appears for the second time, after a preliminary presentation by the chairman of the committee in San Francisco last year, amended in certain details to meet the uniformly friendly suggestions made to the committee by individuals and by societies. It brings with it the unanimous support of every member of the committee, the approval of individuals and societies thruout the country, the tentative indorsement of this Association of the preliminary draft submitted for suggestions last year, and a record of phenomenal interest and discussion in all parts of the United States.

It may seem unnecessary to describe to you at length a report which has received such remarkable distribution and which has been once presented in tentative form by the chairman of the committee. Nor is it my purpose more than to lay before you the salient points which may form a certain basis for discussion, points which have been suggested largely by the widespread discussion of the report thruout the country. As an interesting sidelight upon the unusual interest manifested in this document, I may say that the committee has had the greatest difficulty in securing a sufficient number of copies for use in this meeting, since the edition of five thousand copies authorized this year is now completely exhausted. Of this large number, nearly three thousand have been sent out by the commissioner of education in Washington, each copy to an individual in response to his own direct request, in addition to the two thousand sent out by the committee and the publication of the report by *School Science and Mathematics*.

This demand is important because it demonstrates the real need of such a report, and augurs its future influence and importance. It leads naturally to a statement of the effect which we hope the report will have upon the country. It is not expected or desired that the report should be followed slavishly by anyone, or that what the committee has set down should be regarded as final in its details. It is rather the general spirit of its contents which we hope will receive your approval and which we think will form a basis for further thought and discussion by the individual teacher and by societies thruout the country. It is not the intention that anyone whatever should be bound by any word or phrase; rather it is hoped that the moral effect of the report, backed by the compelling force of its reasonableness and by the general approval of thinking teachers, will carry weight with those who think and debate on questions which it touches. As a stimulus to thought, as a basis for discussion, as a suggestion of one reasonably consistent plan approved by many men of note, it cannot fail to have a vitalizing influence upon the teaching of geometry thruout the entire country. To this end, the committee has stated repeatedly that details are not supposed to be binding or of great importance, that

individual opinion be accorded the widest scope and the greatest respect, that the arbitrary power of examining bodies be curbed, and that schools be free to work out their own plans, using whatever of this report commends itself to their intelligent thought. Such freedom is of the utmost importance; to imagine that this report in any sense limits the freest possible individual action is completely to reverse the progressive tendency which was the keynote of the committee's action; but it is equally true that individual freedom needs and demands some organ for the expression of a general consensus of opinion, some expression of the views of leaders, some basis other than the existing arbitrary demands of examining authorities upon which intelligent plans may be built. It is the purpose of this committee to present to you such a basis, freed from all other considerations than the best teaching of geometry to young students. Your acceptance of it will bind you and other teachers only in so far as the strength of its conclusions is convincing and unanswerable to your minds; it will force the acceptance of its general standards and of the principle of freedom of teaching by all examining bodies; it will lend the great power of your moral support to the further dissemination of the ideas it contains; it will make for progressive discussion and untrammelled advancement in the teaching of this important subject.

The report is not a mere list of theorems. It touches upon every phase of the teaching of geometry. Thus, one of the important portions, placed by the committee at its very front, is an excellent résumé of the history of the teaching of geometry in this country and abroad, prepared largely by Professor Cajori, of the committee, who is well known as an authority on the history of the subject. These historical notes bear vitally on the teaching of the present day. They have been carefully weighed by the committee in preparing the rest of the report. Indeed, the positive effect of historical knowledge of the teaching of the past is so evident that it seems desirable rather to emphasize the fact that the changed conditions of our age and the changes both in the organization of society and in the accepted theories of education make necessary a great degree of caution in accepting without question a position sanctioned by historical facts alone, and render it necessary for us today to review without prejudice experiments of the past, even tho those experiments may have clearly ended in failure in their own day. Thus an attempt to render the teaching of geometry more practical in the Middle Ages, tho unsuccessful, does not discourage the renewal of the same effort now.

The attitude of the report on the question of logical treatment is next emphasized (p. 3). Suffice it here to say that the committee has retained the logical form of discussion, at least in so far as logic means correct reasoning. On the other hand, the extreme formalism too often arbitrarily attached to logic is distinctly abandoned, and the insistence is placed rather upon the students' understanding of the reasoning than upon the satisfaction of ideal standards of pure formalism. The psychological principles which underlie the learning process are everywhere recognized, so that the student is to be gradually led into the realm of logical proof by easy stages, rather than thrown into it in connection with theorems that seem to him self-evident.

Axioms and the axiomatic treatment of the whole subject are retained, but axioms which have no sound psychological basis—such as the so-called “axioms of order”—are excluded. Definitions are called for in a form which will support sound reasoning, but an excess of formal terms, such as *scholium*, and the introduction of new terms, are decried.

Informal proofs of many propositions are sanctioned, at least in many cases in which the theorems are almost self-evident. While the committee has limited itself in this matter more than some teachers desire, the acceptance of the principle of informal proof will prove most acceptable to a great majority of teachers. It is to be understood that propositions of considerable difficulty and propositions of especial importance are not to receive such informal treatment.

The topic of limits and incommensurable cases receives special attention, and it is hoped that the position taken will commend itself as reasonable and as unprejudiced.

While the committee was in thoro agreement regarding these topics, it was felt that a statement which avoided any extreme position might best avoid any appearance of dictation.

The statement regarding the motives for the study of geometry (p. 12) deserves notice. It should be remarked that the old appeal to discipline as a motive has here yielded to other more convincing claims.

Special courses for special classes of students received more consideration than the report would indicate on its face. The final statement (p. 14) expresses the conviction of the committee that these special courses are too varied to permit of detailed suggestions to fit each case which may arise. It is the hope that the arrangement for emphasis of more important theorems will itself solve this difficulty, in offering a sane basis for selection, when a complete course is not possible or desirable.

In making suggestions for geometry in the more elementary grades, the committee outlined possible work of a geometric nature suited to the needs of very young children. Perhaps the greatest insistence needed here is that a clear and emphatic distinction be drawn between geometry on the one hand and logic (i.e., formal logic) on the other hand. Too often these are confused thru their traditional condition; and geometry is supposed to mean logical deduction. It will perhaps be clearest to say that the committee was unanimous and emphatic in opposing any deductive logic whatever in the geometry to be taught in elementary (graded) schools.

Very great attention is paid by the committee to the important question of exercises. A thoro reading of this portion of the report is recommended, since it cannot be reproduced in brief. Concrete problems of a more practical character are urged, and a large number of illustrative exercises are stated. The sources of such problems are discussed, and the reasons for their introduction are stated (p. 19 and p. 20).

Both here, and in the syllabus of theorems, attention is called to the use of algebraic methods of proof, which will tend to insure a higher degree of correlation of these subjects, and which will afford better and simpler means of proof of many theorems and exercises.

Before an attempt is made to read—or even scan—the lists of theorems, it is highly desirable, and indeed absolutely necessary to a correct understanding, to read the preface to the syllabus, pp. 30–33. I may state a few of the fundamental points which might lead to utter misunderstanding if overlooked.

a) The theorems are here stated for the teacher—not in the form most desirable for students. For example, many are written in abbreviated literal notations. Many are combined into triple or multiple statements. The committee distinctly recommends that these be re-worded for students.

b) Distinctions in importance and in corresponding emphasis are indicated by the use of four grades of type. The theorems in black-face type are most important, those in italics next, those in ordinary type next, those in fine type are quite subordinate and verge into those which are not explicitly included in the list. It will be a great mistake, however, to suppose that the committee regards this classification as absolutely final or binding. Some theorems may be a little more, some less, strongly emphasized than would follow from the types here used, at the discretion of the teacher or school. Again, it is the principle, and not the details, for which the committee would strongly contend.

c) No logical order is intended. Any apparent indication of a logical order is unintentional.

d) The principle upon which the theorems are grouped is that of their fundamental connection and dependence, not only logically, but also in their practical applications in mathematics and elsewhere. Here again, any changes whatever desired by anyone will be sanctioned by the committee. Again, it is only the principle for which we contend: that attention be paid in grouping theorems to the fundamental relationships between the theorems placed in the same group, so that a possible basis for a human grasp of the entire field is furnished. Such a grouping, as opposed to an incoherent arrangement which depends only on accidental superficial similarity in topic, is thought to be quite vital.

e) Trigonometric ratios are introduced in their simplest form, as they actually present themselves in geometric figures.

A full discussion of the syllabus, and in fact, of the report as a whole, is expected and earnestly requested by the committee. It would be out of place for me to enter upon a time-consuming exposition of the details of the syllabus, which lies before you in printed form. I may say that the statement on pp. 45-46 gives a clear idea of the view of the committee concerning the treatment of the several grades of theorems.

If, as a result of your discussion here, it appears that there exists a fair consensus of opinion favoring the general principles set forth in the report, we of the committee will feel that our labor has not been in vain. It is neither expected nor desired that all should approve each detail. Such minute agreement is beyond human belief or power. Even the committee does not wish to be understood as standing unanimously for each and every minor phrase, tho there is no statement which has not the unqualified approval of quite a majority, and there is nothing of moment on which the entire committee does not agree in the matter of broad principle. It is on just such a basis that we hope for your approval and support. With it, we can be assured of a hearing by every teacher interested in geometry, we can be assured that the earnest efforts of this committee will not go unheeded, but that they will be given consideration wherever the subject of the teaching of geometry is discussed. Precisely this we do desire—a hearing by all interested in these topics, consideration of what we here present by all who may plan courses or discuss the planning of them. To desire more than a mere hearing not only would violate the freedom which the committee regards as essential to the best growth of the schools of the country, but it would also be a confession of our lack of faith in our own conclusions. To deny that hearing would be just as great a blow at the freedom of the schools, and just as great a confession of weakness by those who deny it. We ask, therefore, your more careful consideration and discussion, to the end that you decide whether you wish to lend the moral support of this body to the further propagation of these ideas and to the stimulation of continued thought and discussion of the teaching of geometry.

DISCUSSION

HENRY W. STAGER, head of the Department of Mathematics, Fresno Junior College, Fresno, Cal.—It needs but a careful reading of the report of our committee to realize its marked excellence. In recent years several associations of mathematics teachers have published syllabi on geometry. While these attempts have been valuable steps in the progress toward a rational standardization of the subject, the present syllabus is the first to have awakened a real national interest, a very essential factor in any advance movement. The personnel of the committee, and the vast amount of work they have expended on the report, remarkable equally for its great liberality and its comprehensive nature, place this syllabus in a position by itself.

Whatever other claims there may be for the study of geometry—and there are many—the chief purpose of this course is to develop sound and accurate *thinking*. That this quality is woefully lacking in our young people needs no reiteration in this gathering. Aside from giving the students a knowledge of the fundamental truths of elementary geometry and arousing in them a genuine liking for the subject, the aim of the teacher should be to select and present the subject-matter so as to bring about the highest development of their thinking power. With this view the attitude of the committee strongly accords, and their report with its well-rounded syllabus keeps the three chief objects constantly in view, and gives to each its proper emphasis.

The feature of the report which appeals most strongly to me is the extreme broad-mindedness of the committee. Thruout they have exercised the greatest care not to appear dogmatic in their attitude. The value of this fair-minded attempt to get at the fundamental truths of geometry in the light both of its historic character and of modern needs cannot be overestimated. Such an attitude demands our highest commendation and augurs well for future progress in the teaching of geometry. Whatever the view-

point of the individual teacher, the report can be used to advantage. For instance, many of us are urging the teaching of mathematics as opposed to the teaching of algebra, geometry, and trigonometry as separate subjects. The committee, and very properly I think, does not consider a discussion of this matter within the province of this report. The syllabus is applicable equally well to either method. Geometry is geometry, whether taught separately or in correlation with other subjects. In this connection, however, I believe that the committee without in any way weakening their position might recommend in their statement regarding entrance examinations that examiners give full credit to work done under the correlation method.

The report opens with an excellent historical introduction and a very valuable bibliography by Professor Florian Cajori. The attitude of the committee (on p. 3) is so admirably expressed that it needs no further comment here, other than again to point out its breadth of view.

The suggestion that the elements of trigonometry be introduced in connection with similar triangles deserves hearty approval. The trigonometric ratios afford not only the means of enlivening an oftentimes very trying section of the course, but also the opportunity for presenting some of the universal applications of geometry. I have noted with much pleasure in various high schools the effect a few lessons in trigonometry have had on the classes. A new interest was at once manifest and the remaining work of the course was carried on with renewed vigor. This work should be introduced even at a sacrifice, but in the average nine and a half or ten months' course, I have found that teachers can usually find the necessary time without omitting any of the essentials.

With the position of the committee on definitions I must take issue. I heartily agree with the general principle that new terms should be introduced and accepted meanings be changed only when the new has a decided advantage over the old and has the general sanction of the mathematical world and not by the action of individual teachers and writers. However, it seems to me that our committee is losing a valuable opportunity to standardize to a large degree the terms and symbols of geometry, a need which every teacher recognizes. The initiative in this movement cannot well be undertaken by any body with greater authority and backing than this committee. And this action must be indefinitely delayed if not undertaken now; for when this report is finally accepted it will probably be many years before a similar task is again undertaken. I would urge that they present a reasonably complete list of those terms and symbols essential to an elementary course, classifying them as far as possible in much the same ways as the theorems have been classified. Naturally such a list would not be exhaustive, but it would undoubtedly tend toward a greater uniformity. The present plan fails to meet the real needs of the situation. For instance, it is suggested that "mixed line" be dropped because it is an antiquated term of no value in elementary geometry. We will all agree to the action, but what similar terms are suggested for dropping by this term? In regard to the symbol for congruence, I would like to urge the symbol $\cong$. As far as possible symbols should be self-explanatory, i.e., they should suggest readily on sight the operation or form they stand for. Congruence, as used in geometry, implies equality in every respect; i.e., in both size and shape, as opposed to equivalence, equality in size or area; and similarity, equality in shape. The symbol suggested, $\cong$, meets the need of indicating this double equality at once, the upper part indicating similarity and the lower part equality of size. The symbol of identity suggested by the committee does not afford this advantage and presents an old symbol to young minds with an interpretation somewhat different from that to which they have been accustomed. This same symbol has a use in geometry, which can best be shown by an illustration. Consider the congruence of the two triangles formed by the median AM of the isosceles triangle BAC . In selecting the three equal parts, it is of value, tho not essential, to say $AM \equiv AM$.

The attitude of the committee on informal proofs and the treatment of limits and

incommensurables will commend itself very strongly to the teacher of the average class. Care must be exercised not to use the informal proof too readily and to apply the method only when the logical proof is beyond the student's knowledge at the time or would not appeal to him as a rigorous proof. The work on limits and incommensurables usually should certainly be confined to explanation and illustration of the underlying principles—and largely illustration. An attempt to prove theorems only leads to hopeless confusion and memory work, which is greatly to be deplored, and is a far greater injury to the real progress of the student than the failure to get a few propositions of doubtful value for an elementary course.

Section C takes up the consideration of special courses. In this connection careful attention must be given to the recommendation for introductory courses in the lower grades. While the needs of the student who concludes his school life with the grammar school are manifest, it is very easy to carry the so-called inventional methods too far and so destroy the real value of the secondary course in geometry. Geometrical drawing is also recommended for the grades. If limited to the use of the ruler and compass in the accurate drawing of simple geometrical forms, some advantage may result. Too often, however, the student acquires a mere mechanical knowledge, which necessitates the greatest effort on the part of the teacher of geometry to direct into the lines of logical construction.

Special courses for special classes of students, as, for instance, those in the manual arts or in agriculture, seem hardly warranted. Such courses are very apt to defeat the very ends claimed for them, and certainly the chief aim of geometry will not be met by any lowering of the requirements to meet the purely utilitarian viewpoint.

Section D presents the question of exercises and problems. The tendency for many years to constantly increase the number of exercises has naturally led to the introduction of many exercises of an abstract character, most of which serve merely as mathematical puzzles for the brighter students. Undoubtedly there should be a large number of concrete applications of the principles of each theorem (most of them of not too great difficulty) in close connection with the propositions involved. These applications should be as practical as possible and in sufficient number for the teacher to select different sets in successive years or for different classes. All exercises should be graded with the greatest care. A carefully selected list, including some of greater difficulty, at the conclusion of a topic would afford the better pupils ample opportunity for the exercise of their ability and for the satisfaction received from accomplishing a difficult task. In my opinion the elementary course presents no opportunity for the abstract mathematical puzzle, unless in a chapter especially devoted to that class of exercises. On the other hand, there is the danger of reaching out to the other extreme of the so-called practical problems. After reading thru many of the texts on applied mathematics for secondary schools, one finds it extremely difficult to select even a few problems which will remotely interest the average student. Such pupils are not interested in the details of architecture, machine design, railroad curves, and the like. Their problems must be of a more fundamental type, intimately associated with the more or less common experiences of all. In this connection, it will be well to recall the value of the trigonometric ratios already suggested. Most students will find pleasure in a method to obtain the height of the flag-pole on the school building, or the distance across the local river, or out to an island in the bay, or the height of a distant mountain. Again, in country districts, the application of areas and perimeters of triangles are numerous. But what interest does the average student find in the confused lines of church windows, or the difficult construction of the railroad curve or the three-centered arch? For instance, of the problems suggested as types by the committee, problem 9, the construction of the three-centered arch, and problem 11, the church window, both on p. 23, fail in a large measure to meet their own requirements and would prove of slight interest to the average student. The figure of the former problem

is misleading and the statement of the problem itself is ambiguous. In both cases the solution is not at all evident, and even when it is indicated as in the figures, the proof needed is too difficult. As far as possible, such problems should be avoided and they certainly should not be given as types.

A valuable source for exercises is found in problems involving loci. With their many applications they afford excellent opportunity for introducing the important concepts of motion and functionality. The committee's suggestion that problems be stated in both forms, "locus of points" and "locus of a point," to bring out clearly two important but different phases of a locus, will meet with no opposition. It applies, in principle, to all classes of exercises and propositions. Wherever possible it is desirable to employ different forms of statement to make clear the fundamental ideas and give facility in expression. The character of the loci problems should be broadened to cover a large class of applications. Almost without exception our textbooks restrict this method to problems involving distance, so much so in fact that students come to define locus in terms of distance. While four of the list of nine illustrations given on p. 28 do not involve distance, I should be glad to see the broader applications of locus emphasized more fully.

When the committee urges the introduction into geometry of algebraic methods and of problems involving algebra it well states that "the interdependence of algebra and geometry is a matter of no small importance both historically and for subsequent mathematical work." The correlation of the two subjects often affords a decided advantage over the use of either alone, and surely there can be no objection when the proofs are clear and rigorous. Again, results which are obtained with difficulty by the use of geometric methods solely may often be readily deduced by a combination of both. Generally there is the additional gain of a new insight into the problem. The construction of the exact square roots of prime numbers by the use of the Pythagorean theorem lends a new interest to the algebraic extraction of square roots. Algebraic formulas constructed geometrically become alive. Where time permits, and sufficient interest can usually make the necessary time, some of the elementary principles of straight lines and circles treated analytically will afford additional means of strengthening the correlation idea.

Section E takes up the syllabus proper. The committee presents its lists of theorems, which it definitely states is not exhaustive, in 21 groups, 15 for plane geometry and 6 for solid geometry. Within each group the theorems are divided into 4 classes according to importance. Certain theorems of each group are printed in black-face type. These theorems are considered fundamental and are to serve as the foundation stones of geometry. Theorems next in importance are printed in italics. A third class is printed in roman type, and a final list of the least important is printed in small type. The average textbook presents about 150 theorems, exclusive of corollaries and the theorems in proportion which are not considered by the committee. The present syllabus consists of 105 theorems; 26 in black-face type, 18 in italics, 34 in roman, and 27 in small type; the last of these the committee suggests be considered largely in the nature of very important exercises, so that the number of required theorems is reduced to 78. We cannot too strongly commend the courage of the committee in this efficient pruning. The modern tendency has been to increase the number of required propositions out of all proportion to their usefulness and without in any way adding to the rigor of the course. Naturally, to those who contend that elementary geometry should be a perfect piece of absolutely rigorous mathematical logic, the action of the committee will appear fatal. On the other hand those who consider this course as the means of giving to young minds a first conception of logical mathematical processes together with certain fundamental truths which will be of value to them in all subsequent work find decided advance in the new syllabus. The latter seems to me the only tenable position. A few things learned thoroly give greater power and more knowledge than many things merely skimmed over. The lesser number of required theorems leaves increased opportunity for the introduction of work

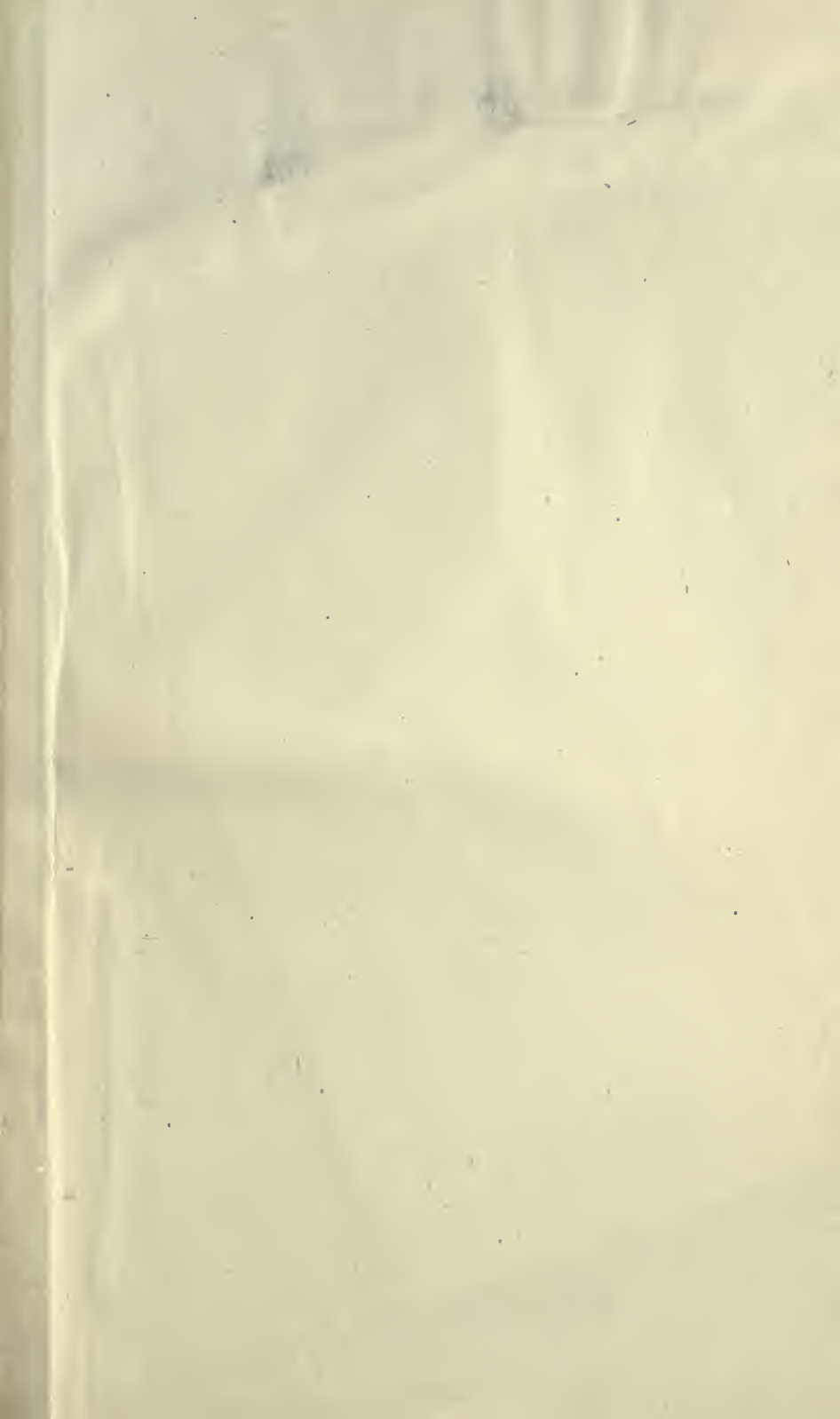
essentially practical, but which cannot now be accomplished because of lack of time, such as the simple applications of the trigonometric ratios. And most important of all, there will now be ample time for original exercises and the development of logical thinking processes. More than all else this faculty of our students needs cultivation and a large number of not too difficult exercises arranged in close connection with the theorems involved afford the best means at our disposal for its development.

Concerning the omissions necessary to make this reduction it would be impossible to secure unanimity of opinion. The committee does not present its lists as final. I regret to find among the omissions such theorems as the centroid of a triangle, the Golden Section, and Hero's formula for the area of a triangle. All of these play an actual and valuable part in the subsequent work of the student. The first and third are very useful, whether he goes out into life or continues his mathematics, while the historical associations of the Golden Section and its relation to the construction of the regular pentagon and the five-pointed star should give it a place. Theorems of this type should not be included in the course at the discretion of the teacher.

I should also be glad to see the committee place more emphasis on proportion. Its treatment should be carefully outlined and not be left to the work in algebra. For the examining board in some states, California for instance, does not include ratio and proportion as a part of the first year's course in algebra and consequently the subject is usually first considered in geometry. A demand for a purely geometrical treatment of proportion in an elementary course seems hardly to be warranted. The algebraic treatment, correlated to the ideas of line-segments as far as possible after the manner suggested for other geometrical concepts in correlation with algebra, would best meet the needs of elementary geometry.

At the close of the list of theorems, the committee suggests a method by which the syllabus may serve as a criterion on the part of the college examiner and the teacher. The basis of the recommendation is emphasis, depending upon the importance of the four types of theorems, and is in full accord with the spirit of the report.

From the standpoint of the teachers of geometry, especially those of less experience, the arrangement by emphasis will prove of greatest value. This offers an outlook over the entire subject gained only by long experience and an extensive knowledge of mathematics. If properly applied in future texts, it will serve to break the monotony which now discourages the student because it affords him no opportunity of evaluating the work he is doing. This emphasis, with a topical arrangement so far as it is consistent with logical order, will mean new life in the classroom, a more thoro knowledge of geometry, and increased power on the part of the student.



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